# Supersymmetric $U(N)$ Gauge Model and Partial Breaking of $\mathcal{N} = 2$ Supersymmetry

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## Abstract

Guided by the gauging of $U(N)$ isometry associated with the special Kähler geometry, and the discrete $R$ symmetry, we construct the $\mathcal{N} = 2$ supersymmetric action of a $U(N)$ invariant nonabelian gauge model in which rigid $\mathcal{N} = 2$ supersymmetry is spontaneously broken to $\mathcal{N} = 1$. This generalizes the abelian model considered by Antoniadis, Partouche and Taylor. We shed light on complexity of the supercurrents of our model associated with a broken $\mathcal{N} = 2$ supermultiplet of currents, and discuss the spontaneously broken supersymmetry as an approximate fermionic shift symmetry.

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# I. Introduction

Continuing investigations have been made for more than two decades on supersymmetric field theories, [*] hoping to obtain realistic description of nature by broken $\mathcal{N} = 1$ supersymmetry at an observable energy scale. On the other hand, it is most natural to view that physics beyond this energy scale is controlled by string theory, which, without nontoroidal backgrounds, produces extended supersymmetries in four dimensions. Breaking of extended supersymmetries in this vein provides a bridge between gauge field theory and string theory. String theory does not possess genuine coupling constants: instead, they are the vacuum expectation values of some supersymmetry preserving moduli fields. We are thus led to search for the possibility of spontaneous partial breaking of extended supersymmetries in four dimensions.

In the context of $\mathcal{N} = 2$ supergravity [4], spontaneous breaking of local $\mathcal{N} = 2$ supersymmetry to its $\mathcal{N} = 1$ counterpart has been accomplished by the simultaneous realization of the Higgs and the super Higgs mechanisms. Sizable amount of literature has been accumulated till today along this direction [5, 6, 7]. There have been active researches carried out on nonlinear realization of extended supersymmetries in the partially broken phase [8, 9, 10, 11, 12, 13]. These are closely related to the effective description of string theory [14], brane dynamics [15, 16, 17, 18, 19, 20] and domain walls [21].

After [8, 9] and prior to the remainder of the works on nonlinear realization, there was a work within the linear realization done by Antoniadis, Partouche and Taylor [22] who constructed an $\mathcal{N} = 2$ supersymmetric, self-interacting $U(1)$ model with one (or several) abelian $\mathcal{N} = 2$ vector multiplet(s) [23] which breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$ spontaneously. See also [24, 25]. The partial breaking of supersymmetry is accomplished by the simultaneous presence of the electric and magnetic Fayet-Iliopoulos terms, which is a generalization of [26]. In the present paper, generalizing the work of [22], we construct the $\mathcal{N} = 2$ supersymmetric action of a $U(N)$ invariant nonabelian gauge model in which rigid $\mathcal{N} = 2$ supersymmetry is spontaneously broken to $\mathcal{N} = 1$. The gauging of $U(N)$ isometry associated with the special Kähler geometry, and the discrete $R$ symmetry are the primary ingredients of our construction.

Let us recall that partial breaking of extended rigid supersymmetries appears not possible on the basis of the positivity of the supersymmetry charge algebra:

$$\left\{ \bar{Q}^i_\alpha, Q_{j\dot\alpha} \right\} = 2(\mathbf{1})_{\alpha\dot\alpha} \delta^i{}_j H. \tag{1.1}$$

In fact, if $Q_1|0\rangle = 0$, one concludes $H|0\rangle = 0$ and $Q_i|0\rangle = 0$ for all $i$. If $Q_1|0\rangle \neq 0$, then $H|0\rangle = E|0\rangle$ with $E > 0$ and $Q_i|0\rangle \neq 0$ for all $i$. The loophole to this argument is that the

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[*]See [1, 2, 3] to review

2

use of the local version of the charge algebra is more appropriate in spontaneously broken symmetries and the most general supercurrent algebra is

$$\{\bar{Q}^j_{\dot{\alpha}}, \mathcal{S}^m_{\alpha i}(x)\} = 2(\sigma^n)_{\alpha\dot{\alpha}}\delta^j_{\;i}T^m_n(x) + (\sigma^m)_{\alpha\dot{\alpha}}C^j_i, \tag{1.2}$$

where $\mathcal{S}^m_{\alpha i}$ and $T^m_n$ are the supercurrents and the energy momentum tensor respectively. We have denoted by $C^j_i$ a field independent constant matrix permitted by the constraints from the Jacobi identity [27]. This last term does not modify the supersymmetry algebra acting on the fields. The abelian model of [22] and our nonabelian generalization provide a concrete example of this local algebra within linear realization from the point of view of the action principle.

The Lagrangian of our model has noncanonical kinetic terms coming from the nontrivial Kähler potential and does not fall into the class of renormalizable Lagrangians. As a model with spontaneously broken $\mathcal{N} = 2$ supersymmetry, the prepotential $\mathcal{F}$ is present from the beginning of our construction. This is in contrast with breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by the operator (superpotential) $W(\Phi)$, where $\mathcal{F}$ appears aposteriori according to the recent developments beginning with Dijkgraaf and Vafa [28]. The model has a $U(1)$ sector interacting with an $SU(N)$ sector and the spontaneously broken supersymmetry acts as an approximate fermionic shift symmetry. Piecing through all these properties, we conclude that the action of the model should be regarded as a low energy effective action which applies to various processes and that the dynamical effects including those of (fractional) instantons are to be contained in the prepotential as an input. This input should be supplied by a separate means of calculation. The connection with the exact determination of the prepotential via [29, 30] and from integrable systems [31] [32] offers a new avenue of thoughts with this regard.

In section II, we provide the construction of the $\mathcal{N} = 2$ supersymmetric action of the $U(N)$ invariant nonabelian gauge model which is equipped with the Fayet-Iliopoulos $D$ term and a specific superpotential. Gauging of the noncanonical kinetic terms coming from the Kähler potential is a necessary step to complete the action. In section III, we provide the transformation law of the extended supersymmetries associated with the model. We note that the $SU(2)$ automorphism of $\mathcal{N} = 2$ supersymmetry has been fixed in the parameter space. In section IV, we fix the form of the prepotential and determine the vacuum with unbroken gauge symmetry. We exhibit partial breaking of $\mathcal{N} = 2$ supersymmetry and discuss a mechanism which enables this. In section V, we examine a broken $\mathcal{N} = 2$ supermultiplet of currents [33] associated with the model. The $U(1)_R$ current is not conserved except for the case where the prepotential has an $R$-weight two. Despite this, we show that the broken $\mathcal{N} = 2$ supermultiplet of currents provides a useful means to construct the extended supercurrents. We shed light upon their complexity. In section VI, we discuss a role played by the spontaneously broken supersymmetry. We see that it acts as a approximate $U(1)$ fermionic shift symmetry in the limit of letting the magnetic Fayet-Iliopoulos term large

relative to the electric one. Our discussion in section two and that in section three leading to $\mathcal{N} = 2$ supersymmetric Lagrangian exploit an algebraic operation denoted by $\mathfrak{R}$. This operation is defined by including the sign flip of the Fayet-Iliopoulos parameter $\xi \to -\xi$ into the standard discrete canonical transformation $R$. It is a legitimate algebraic process to use $\mathfrak{R}$ to demonstrate the second supersymmetry and in section three we obtain $\mathcal{N} = 2$ supersymmetry transformation by demanding the covariance under $\mathfrak{R}$. In Appendix A, we give a more pedagogical proof of $\mathcal{N} = 2$ supersymmetry of our action, using the canonical $R$. The two approaches are thus shown to be equivalent. In Appendix B, we reexamine the $\mathcal{N} = 1$ current supermultiplet [34] in the Wess-Zumino model.

## II.   $\mathcal{N} = 2$ $U(N)$ Gauge Model

Let us first state our strategy to obtain the $\mathcal{N} = 2$ supersymmetric action with nonabelian $U(N)$ gauge symmetry. We adopt the $\mathcal{N} = 1$ superspace formalism to write down a $U(N)$ invariant action consisting of a set of $\mathcal{N} = 1$ $U(N)$ chiral superfields and vector superfields in the adjoint representation. The action at this level is equipped with the terms required for the gauging, the Fayet-Iliopoulos D term, and a generic superpotential. Imposing the discrete element of $SU(2)$ automorphism of $\mathcal{N} = 2$ supersymmety algebra as symmetry of our action [2, 22], we obtain the action mentioned in the introduction.

What is meant by this last procedure is, however, a little more subtle than one might first think and we pause to explain this here in more detail. In the presence of the Fayet-Iliopoulos D term with its coefficient $\xi$, $\mathcal{N} = 1$ Lagrangian is in general not invariant under the discrete $R$ symmetry. (See (2.39)). Best one can do is therefore to consider simultaneously an inversion of the parameter $\xi$. (See (2.49)). Under this extended operation denoted by $\mathfrak{R}$, we will find

$$\mathfrak{R} : \mathcal{L} \to \mathcal{L}, \quad \mathfrak{R} : \mathcal{L}' \to \mathcal{L}'. \tag{2.1}$$

(See (2.26), (2.33).) Combining this with the algebra

$$\mathfrak{R}\delta_1\mathfrak{R}^{-1} = \delta_2, \tag{2.2}$$

we conclude that our final actions (2.33) and (2.64) with (2.45) and (2.48) are invariant under $\mathcal{N} = 2$ supersymmetry. Here we denote by $\delta_1$ and $\delta_2$, the transformation of the first supersymmetry and that of the second supersymmetry respectively. This definition $\mathfrak{R}$ turns out to be consistent with an interpretation that full rigid $SU(2)$ symmetry has been fixed in the parameter space. This is discussed in section III.

4

## A. $U(N)$ **Gauge Model**

Let us introduce a set of $\mathcal{N} = 1$ chiral superfields

$$\Phi(x^m, \theta) = \sum_{a=0}^{N^2-1} \Phi^a t_a \,. \tag{2.3}$$

Here, $t_a$, $a = 0, 1, \ldots, (N^2 - 1)$, are $N \times N$ hermitian matrices which generate $u(N)$ algebra, and $t_{\hat{a}}$, $\hat{a} = 1, \ldots, (N^2 - 1)$, generate $su(N)$ algebra

$$[t_{\hat{a}}, t_{\hat{b}}] = i f_{\hat{a}\hat{b}}^{\hat{c}} t_{\hat{c}}. \tag{2.4}$$

The index 0 refers to the overall $u(1)$ generator. The scalar fields $A = A^a t_a$ in $\Phi$ undergo the adjoint action

$$A \to UAU^\dagger, \tag{2.5}$$

under $U(N)$.

The kinetic term for $A$ is generated by

$$\mathcal{L}_K = \int d^2\theta d^2\bar{\theta} \; K(\Phi^a, \Phi^{*a}), \tag{2.6}$$

where $K(A^a, A^{*a})$ is the Kähler potential. The Kähler potential we employ is given by

$$K(A^a, A^{*a}) = \frac{i}{2}(A^a \mathcal{F}_a^* - A^{*a} \mathcal{F}_a), \tag{2.7}$$

where $\mathcal{F}_a = \partial_a \mathcal{F} = \frac{d}{dA^a}\mathcal{F}$ and $\mathcal{F}$ is an analytic function of $A$.$^\star$ The Kähler potential can be written using a hermitian metric on the bundle compatible with the symplectic structure as

$$K = -\frac{i}{2}\left\langle \Omega | \bar{\Omega} \right\rangle, \quad \left\langle \Omega | \bar{\Omega} \right\rangle = -\Omega^T \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \Omega^* \,. \tag{2.8}$$

The Kähler metric

$$g_{ab^*} = \partial_a \partial_{b^*} K = \mathrm{Im}\,\mathcal{F}_{ab} \tag{2.9}$$

constructed this way always admits a $U(N)$ isometry. The holomorphic Killing vectors $k_a = k_a{}^b \partial_b$ are generated by the Killing potential $\mathfrak{D}_a$, to be introduced shortly, as

$$k_a{}^b = -i g^{bc^*} \partial_{c^*} \mathfrak{D}_a, \quad k_a^{*b} = i g^{cb^*} \partial_c \mathfrak{D}_a. \tag{2.10}$$

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$^\star$The $\Omega = \binom{A^a}{\mathcal{F}_b}$ can be regarded as a section of a holomorphic symplectic bundle on a special Kähler geometry (see [34] and references therein). We work in special coordinates in this paper.

These form an algebra $[k_a, k_b] = -f^c_{ab} k_c$. The $A^a$ and $\mathcal{F}_a$ transform in the adjoint representation of $U(N)$

$$\delta_b A^a = -f^a_{bc} A^c, \qquad \delta_b \mathcal{F}_a = -f^c_{ab} \mathcal{F}_c . \tag{2.11}$$

One finds that the commutator of two of $\delta_a$ is given by $[\delta_a, \delta_b] = f^c_{ab} \delta_c$. Comparing this with the commutator of two Killing vectors, we are able to identify $\delta_a$ with $-k_a$. The equation (2.11) is rewritten as

$$k_b{}^c \partial_c A^a = f^a_{bc} A^c, \qquad k_b{}^c \partial_c \mathcal{F}_a = -f^c_{ba} \mathcal{F}_c . \tag{2.12}$$

The isometry group can be embedded in the symplectic group, and the $\mathfrak{D}_a$ is given by

$$\mathfrak{D}_a = -\frac{1}{2} \langle \Omega | T_a \bar{\Omega} \rangle = -\frac{1}{2} (\mathcal{F}_b f^b_{ac} A^{*c} + \mathcal{F}_b^* f^b_{ac} A^c), \qquad T_a = \begin{pmatrix} f^b_{ac} & 0 \\ 0 & -f^b_{ac} \end{pmatrix}. \tag{2.13}$$

Note that $\mathfrak{D}_{\hat{a}}$ are completely determined by this formula while $\mathfrak{D}_0$ is determined up to a constant.

In order to gauge the $U(N)$ isometry, we introduce a set of $\mathcal{N} = 1$ vector superfields

$$V(x^m, \theta, \bar{\theta}) = \sum_{a=0}^{N^2-1} V^a t_a. \tag{2.14}$$

The $U(N)$ gauging of $\mathcal{L}_K$ is accomplished [35] by adding

$$\mathcal{L}_\Gamma = \int d^2\theta d^2\bar{\theta} \, \Gamma, \qquad \Gamma = \left[ \int_0^1 d\alpha \, e^{\frac{i}{2}\alpha v^a (k_a - k_a^*)} v^c \mathfrak{D}_c \right]_{v^a \to V^a}, \tag{2.15}$$

where $[\cdots]_{v^a \to V^a}$ means the replacement of $v^a$ by $V^a$ after evaluating $\cdots$. Combining $\mathcal{L}_K$ with $\mathcal{L}_\Gamma$, we obtain

$$\begin{aligned} \mathcal{L}_K + \mathcal{L}_\Gamma &= -g_{ab^*} \mathcal{D}_m A^a \mathcal{D}^m A^{*b} - \frac{i}{2} g_{ab^*} \psi^a \sigma^m \mathcal{D}'_m \bar{\psi}^b + \frac{i}{2} g_{ab^*} \mathcal{D}'_m \psi^a \sigma^m \bar{\psi}^b \\ &\quad + g_{ab^*} F^a F^{*b} - \frac{1}{2} g_{ab^*,c^*} F^a \bar{\psi}^b \bar{\psi}^c - \frac{1}{2} g_{bc^*,a} F^{*c} \psi^a \psi^b \\ &\quad + \frac{1}{\sqrt{2}} g_{ab^*} (\lambda^c \psi^a k_c^{*b} + \bar{\lambda}^c \bar{\psi}^b k_c{}^a) + \frac{1}{2} D^a \mathfrak{D}_a , \end{aligned} \tag{2.16}$$

where we have exploited $\frac{1}{4} g_{ac^*,bd^*} \psi^a \psi^b \bar{\psi}^c \bar{\psi}^d = 0$ as $g_{ac^*,bd^*} = 0$ for the choice of $K$ in (2.7). The covariant derivatives are defined as

$$\mathcal{D}_m A^a = \partial_m A^a - \frac{1}{2} v^b_m k_b{}^a, \tag{2.17}$$

$$\mathcal{D}'_m \psi^a = \mathcal{D}_m \psi^a + \Gamma^a_{bc} \mathcal{D}_m A^b \psi^c, \tag{2.18}$$

$$\mathcal{D}_m \psi^a = \partial_m \psi^a - \frac{1}{2} v^b_m \partial_c k_b{}^a \psi^c , \tag{2.19}$$

6

where $\Gamma^a_{bc} = g^{ad^*} g_{bd^*,c}$.

The gauged kinetic action for the vector superfield $V$ is given by

$$\mathcal{L}_{\mathcal{W}^2} = -\frac{i}{4} \int d^2\theta \tau_{ab} \mathcal{W}^a \mathcal{W}^b + c.c. , \quad \mathcal{W}_\alpha = -\frac{1}{4} \bar{D}\bar{D} e^{-V} D_\alpha e^V = \mathcal{W}^a_\alpha t_a , \quad (2.20)$$

where $\tau_{ab} = (\tau_1)_{ab} + i(\tau_2)_{ab}$ is an analytic function of $\Phi$, and will be determined by requiring $\mathcal{N} = 2$ supersymmetry. The $\mathcal{L}_{\mathcal{W}^2}$ is evaluated as

$$\mathcal{L}_{\mathcal{W}^2} = -\frac{1}{2}\tau_{ab}\lambda^a\sigma^m\mathcal{D}_m\bar{\lambda}^b - \frac{1}{2}\bar{\tau}_{ab}\mathcal{D}_m\lambda^a\sigma^m\bar{\lambda}^b - \frac{1}{4}(\tau_2)_{ab}v^a_{mn}v^{bmn} - \frac{1}{8}(\tau_1)_{ab}\epsilon^{mnpq}v^a_{mn}v^b_{pq}$$

$$-i\frac{\sqrt{2}}{8}(\partial_c\tau_{ab}\psi^c\sigma^n\bar{\sigma}^m\lambda^a - \partial_{c^*}\tau^*_{ab}\bar{\lambda}^a\bar{\sigma}^m\sigma^n\bar{\psi}^c)v^b_{mn}$$

$$+\frac{1}{2}(\tau_2)_{ab}D^aD^b + \frac{\sqrt{2}}{4}(\partial_c\tau_{ab}\psi^c\lambda^a + \partial_{c^*}\tau^*_{ab}\bar{\psi}^c\bar{\lambda}^a)D^b + \frac{i}{4}\partial_c\tau_{ab}F^c\lambda^a\lambda^b - \frac{i}{4}\partial_{c^*}\tau^*_{ab}F^{*c}\bar{\lambda}^a\bar{\lambda}^b$$

$$-\frac{i}{8}\partial_c\partial_d\tau_{ab}\psi^c\psi^d\lambda^a\lambda^b + \frac{i}{8}\partial_{c^*}\partial_{d^*}\tau^*_{ab}\bar{\psi}^c\bar{\psi}^d\bar{\lambda}^a\bar{\lambda}^b, \quad (2.21)$$

where we have defined

$$v^a_{mn} = \partial_m v^a_n - \partial_n v^a_m - \frac{1}{2}f^a_{bc}v^b_m v^c_n, \quad (2.22)$$

$$\mathcal{D}_m\lambda^a = \partial_m\lambda^a - \frac{1}{2}f^a_{bc}v^b_m\lambda^c. \quad (2.23)$$

In addition, we include the superpotential term

$$\mathcal{L}_W = \int d^2\theta W(\Phi) + c.c.$$

$$= F^a\partial_a W - \frac{1}{2}\partial_a\partial_b W\psi^a\psi^b + F^{*a}\partial_{a^*}W^* - \frac{1}{2}\partial_{a^*}\partial_{b^*}W^*\bar{\psi}^a\bar{\psi}^b , \quad (2.24)$$

and the Fayet-Iliopoulos D-term [26]

$$\mathcal{L}_D = \xi \int d^2\theta d^2\bar{\theta} V^0 = \sqrt{2}\xi D^0. \quad (2.25)$$

The superpotential $W$ will be determined by requiring $\mathcal{N} = 2$ supersymmetry. Finally, putting all these together, the total action is given as

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_\Gamma + \mathcal{L}_{\mathcal{W}^2} + \mathcal{L}_W + \mathcal{L}_D. \quad (2.26)$$

For the sake of our discussion in the next subsection, we present the on-shell action, eliminating the auxiliary fields by using the equations of motion

$$D^a = \hat{D}^a - (\tau_2^{-1})^{ab}\left(\frac{1}{2}\mathfrak{D}_b + \sqrt{2}\xi\delta_b^0\right), \tag{2.27}$$

$$F^a = \hat{F}^a - g^{ab^*}\partial_{b^*}W^*, \tag{2.28}$$

$$F^{*a} = \hat{F}^{*a} - g^{ba^*}\partial_b W, \tag{2.29}$$

where

$$\hat{D}^a = -\frac{\sqrt{2}}{4}(\tau_2^{-1})^{ab}\left(\partial_d\tau_{bc}\psi^d\lambda^c + \partial_{d^*}\tau_{bc}^*\bar\psi^d\bar\lambda^c\right), \tag{2.30}$$

$$\hat{F}^a = -g^{ab^*}\left(-\frac{i}{4}\partial_{b^*}\tau_{cd}^*\bar\lambda^c\bar\lambda^d - \frac{1}{2}g_{cb^*,d}\psi^c\psi^d\right), \tag{2.31}$$

$$\hat{F}^{*a} = -g^{ba^*}\left(\frac{i}{4}\partial_b\tau_{cd}\lambda^c\lambda^d - \frac{1}{2}g_{bc^*,d^*}\bar\psi^c\bar\psi^d\right). \tag{2.32}$$

The action $\mathcal{L}$ takes the following form;

$$\mathcal{L}' = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{fermi}^4} \tag{2.33}$$

where

$$\mathcal{L}_{\text{kin}} = -g_{ab^*}\mathcal{D}_m A^a \mathcal{D}^m A^{*b} - \frac{1}{4}(\tau_2)_{ab}v_{mn}^a v^{bmn} - \frac{1}{8}(\tau_1)_{ab}\epsilon^{mnpq}v_{mn}^a v_{pq}^b \tag{2.34}$$

$$-\frac{1}{2}\tau_{ab}\lambda^a\sigma^m\mathcal{D}_m\bar\lambda^b - \frac{1}{2}\tau_{ab}^*\mathcal{D}_m\lambda^a\sigma^m\bar\lambda^b - \frac{i}{2}g_{ab^*}\psi^a\sigma^m\mathcal{D}_m\bar\psi^b + \frac{i}{2}g_{ab^*}\mathcal{D}_m\psi^a\sigma^m\bar\psi^b,$$

$$\mathcal{L}_{\text{pot}} = -\frac{1}{2}\left(\tau_2^{-1}\right)^{ab}\left(\frac{1}{2}\mathfrak{D}_a + \sqrt{2}\xi\delta_a^0\right)\left(\frac{1}{2}\mathfrak{D}_b + \sqrt{2}\xi\delta_b^0\right) - g^{ab^*}\partial_a W\partial_{b^*}W^*, \tag{2.35}$$

$$\mathcal{L}_{\text{Pauli}} = -i\frac{\sqrt{2}}{8}\partial_c\tau_{ab}\psi^c\sigma^n\bar\sigma^m\lambda^a v_{mn}^b + i\frac{\sqrt{2}}{8}\partial_{c^*}\tau_{ab}^*\bar\lambda^a\bar\sigma^m\sigma^n\bar\psi^c v_{mn}^b, \tag{2.36}$$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\partial_a\partial_b W\psi^a\psi^b - g^{ab^*}\partial_a W\left(-\frac{i}{4}\partial_{b^*}\tau_{cd}^*\bar\lambda^c\bar\lambda^d - \frac{1}{2}g_{cb^*,d}\psi^c\psi^d\right)$$

$$-\frac{1}{2}\partial_{a^*}\partial_{b^*}W^*\bar\psi^a\bar\psi^b - g^{ab^*}\left(\frac{i}{4}\partial_a\tau_{cd}\lambda^c\lambda^d - \frac{1}{2}g_{ac^*,d^*}\bar\psi^c\bar\psi^d\right)\partial_{b^*}W^*$$

$$+\frac{1}{\sqrt{2}}g_{ab^*}\left(\bar\lambda^c\bar\psi^b k_c{}^a + \lambda^c\psi^a k_c^{*b}\right)$$

$$-\frac{\sqrt{2}}{4}\left(\tau_2^{-1}\right)^{ab}\left(\frac{1}{2}\mathfrak{D}_a + \sqrt{2}\xi\delta_a^0\right)\left(\partial_d\tau_{bc}\psi^d\lambda^c + \partial_{d^*}\tau_{bc}^*\bar\psi^d\bar\lambda^c\right), \tag{2.37}$$

$$\mathcal{L}_{\text{fermi}^4} = -\frac{i}{8}\partial_c\partial_d\tau_{ab}\psi^c\psi^d\lambda^a\lambda^b + \frac{i}{8}\partial_{c^*}\partial_{d^*}\tau_{ab}^*\bar\psi^c\bar\psi^d\bar\lambda^a\bar\lambda^b$$

$$-\frac{1}{16}\left(\tau_2^{-1}\right)^{ab}\left(\partial_d\tau_{ac}\psi^d\lambda^c + \partial_{d^*}\tau_{ac}^*\bar\psi^d\bar\lambda^c\right)\left(\partial_f\tau_{be}\psi^f\lambda^e + \partial_{f^*}\tau_{be}^*\bar\psi^f\bar\lambda^e\right)$$

$$-g^{ab^*}\left(\frac{i}{4}\partial_a\tau_{cd}\lambda^c\lambda^d - \frac{1}{2}g_{ac^*,d^*}\bar\psi^c\bar\psi^d\right)\left(-\frac{i}{4}\partial_{b^*}\tau_{ef}^*\bar\lambda^e\bar\lambda^f - \frac{1}{2}g_{eb^*,f}\psi^e\psi^f\right). \tag{2.38}$$

8

## B. Discrete $R$-symmetry

We shall show that our Lagrangian (2.33) $\mathcal{L}'$ can be made invariant under the action

$$\mathfrak{R} : \begin{pmatrix} \lambda^a \\ \psi^a \end{pmatrix} \rightarrow \begin{pmatrix} \psi^a \\ -\lambda^a \end{pmatrix}, \tag{2.39}$$

which is a discrete element of the $SU(2)$ $R$-symmetry that acts as an automorphism of $\mathcal{N} = 2$ supersymmetry.

First, we examine the invariance of $\mathcal{L}_{\text{Pauli}}$, $\mathcal{L}_{\text{fermi}^4}$ and $\mathcal{L}_{\text{kin}}$ under the action (2.39). The invariance of $\mathcal{L}_{\text{Pauli}}$ and that of $\mathcal{L}_{\text{fermi}^4}$ under (2.39) require

$$\partial_c \tau_{ab} = \partial_a \tau_{cb}, \tag{2.40}$$

and

$$\partial_c \partial_d \tau_{ab} = \partial_a \partial_b \tau_{cd} , \quad \partial_c \tau_{ab} = \mathcal{F}_{abc}, \tag{2.41}$$

respectively. In addition, the invariance of the fermion kinetic terms in $\mathcal{L}_{\text{kin}}$ implies that

$$\text{Im}(\tau_{ab}) = \text{Im}(\mathcal{F}_{ab}) \tag{2.42}$$

and

$$-2\partial_a \partial_{b^*} \mathfrak{D}_c = \tau_{ad} f_{cb}^d + \tau_{bd}^* f_{ca}^d, \tag{2.43}$$

as well as the last condition in (2.41) which comes from that the terms with a derivative of $A^*$ vanish. The first condition (2.42) comes from the terms with a derivative of $\lambda$ or $\psi$ while the second one (2.43) from those including $v_m^a$. For the boson kinetic terms in $\mathcal{L}_{\text{kin}}$, the invariance is obvious because they do not contain fermionic fields. From the conditions (2.41) and (2.42), we conclude that

$$\tau_{ab} = \mathcal{F}_{ab}, \tag{2.44}$$

so that $g_{ab^*} = (\tau_2)_{ab}$. It is easy to show that the Killing potential $\mathfrak{D}_a$ defined in (2.13) solves the condition (2.43).

Secondly, we examine the invariance of the $\lambda\lambda$ and $\psi\psi$ mass terms in $\mathcal{L}_{\text{mass}}$ under (2.39). The key relation required for this invariance is

$$-\frac{i}{4} g^{cd^*} \partial_c \tau_{ab} \partial_{d^*} W^* = \frac{1}{2} g^{cd^*} \partial_c W \, g_{ad^*,b} - \frac{1}{2} \partial_a \partial_b W. \tag{2.45}$$

Writing the $U(N)$ invariant function $W$ as $W = eA^0 + m\phi(A)$, where the $e$ and $m$ are real constants, it reduces to

$$\mathcal{F}_{abc} (\frac{1}{\mathcal{F} - \mathcal{F}^*})^{cd} (\partial_d \phi - \partial_{d^*} \phi^*) = \partial_a \partial_b \phi, \tag{2.46}$$

which can be solved by $\phi = \mathcal{F}_0 + const$. Thus we can choose

$$W = eA^0 + m\mathcal{F}_0, \tag{2.47}$$

up to an irrelevant constant.

Thirdly, we examine the $\psi\lambda$ terms in $\mathcal{L}_{\text{mass}}$. Because $\psi^a\lambda^b$ is odd under the action (2.39), the coefficient, $\frac{1}{\sqrt{2}}g_{ac^*}k_b^{*c} - \frac{\sqrt{2}}{8}(\tau_2^{-1})^{cd}\partial_a\tau_{cb}(\mathfrak{D}_d + 2\sqrt{2}\xi\delta_d^0)$, must be odd. This implies the key relation for the invariance

$$i\partial_a\mathfrak{D}_b + i\partial_b\mathfrak{D}_a - \frac{1}{2}(\tau_2^{-1})^{cd}\partial_a\tau_{cb}\mathfrak{D}_d = 0, \tag{2.48}$$

as well as

$$\mathfrak{R}; \ \xi \to -\xi. \tag{2.49}$$

The equation (2.48) can be proven as follows. First, we note that

$$\mathcal{F}_{ac}f_{db}^c + \mathcal{F}_{bc}f_{da}^c = -\mathcal{F}_{abc}f_{de}^c A^e, \tag{2.50}$$

which is derived as a derivative of the second relation in (2.12). Using this relation and the definition (2.13), one finds that

$$i\partial_a\mathfrak{D}_b + i\partial_b\mathfrak{D}_a = -\frac{i}{2}\mathcal{F}_{abc}f_{de}^c A^{*d}A^e. \tag{2.51}$$

On the other hand, the Killing potential is shown to be rewritten as

$$\mathfrak{D}_a = \frac{1}{2}f_{cd}^b A^{*c}A^d(\mathcal{F}_{ab}^* - \mathcal{F}_{ab}) = -ig_{ab}f_{cd}^b A^{*c}A^d \tag{2.52}$$

by using the second relation in (2.12). The equations (2.51) and (2.52) are enough to see that the equation (2.48) is true.

Lastly, we examine $\mathcal{L}_{\text{pot}}$. The invariance of $\mathcal{L}_{\text{pot}}$ under (2.49) follows from the fact that the term linear in $\xi$ in $\mathcal{L}_{\text{pot}}$ vanishes:

$$-\frac{1}{2}(\tau_2^{-1})^{ab}\mathfrak{D}_a\sqrt{2}\xi\delta_b^0 = -\frac{\sqrt{2}}{2}\xi g^{a0}(-ig_{ab}f_{cd}^b A^{*c}A^d) = i\frac{\sqrt{2}}{2}\xi f_{cd}^0 A^{*c}A^d = 0 \tag{2.53}$$

where we have used (2.44) and (2.52).

In summary, we have shown that our on-shell action (2.33) admits the discrete $\mathfrak{R}$-symmetry (2.39) and (2.49) if we choose $\tau_{ab}$ as (2.44) and $W$ as (2.47).

We will show that the discrete $\mathfrak{R}$-symmetry can be realized in the off-shell action (2.26) with (2.44) and (2.47). In an ungauged theory without a superpotential, the discrete action on the auxiliary fields is $D^a \to -D^a$ and $F^a \to F^{*a}$. In our model, this is modified as is seen

10

below. The terms which need to be checked are those including auxiliary fields. First, we examine bosonic terms including $F^a$ and $F^{*a}$,

$$g_{ab^*} F^a F^{*b} + F^a \partial_a W + F^{*a} \partial_{a^*} W^*. \tag{2.54}$$

Apparently, this is not invariant under $F \to F^*$. Rewriting it as

$$g_{ab^*}(F^a + g^{ac^*} \partial_{c^*} W^*)(F^{*b} + g^{db^*} \partial_d W) - g^{ab^*} \partial_a W \partial_{b^*} W^*, \tag{2.55}$$

one finds that the action

$$\mathfrak{R} : F^a + g^{ac^*} \partial_{c^*} W^* \to F^{*b} + g^{db^*} \partial_d W \tag{2.56}$$

is a symmetry. Secondly, we consider the $\psi\psi$ and $\lambda\lambda$ mass terms in (2.16), (2.24) and (2.21). Under the action (2.39) and (2.56) the $\psi\psi$ mass terms become

$$\left( \frac{i}{4} \mathcal{F}_{abc}(F^c + g^{cd^*} \partial_{d^*} W^*) - \frac{i}{4} \mathcal{F}_{abc} g^{dc^*} \partial_d W - \frac{1}{2} \partial_a \partial_b W \right) \lambda^a \lambda^b. \tag{2.57}$$

Equating it with the original $\lambda\lambda$ mass term, $\frac{i}{4} \partial_c \tau_{ab} F^c \lambda^a \lambda^b$, we find that the invariance implies

$$\frac{i}{4} \mathcal{F}_{abc}(g^{cd^*} \partial_{d*} W^* - g^{dc^*} \partial_d W) - \frac{1}{2} \partial_a \partial_b W = 0. \tag{2.58}$$

It is easy to see that the superpotential (2.47) solves this equation. Thirdly, we examine the $\psi\lambda$ mass term in (2.16) and (2.21)

$$\frac{1}{\sqrt{2}}(g_{ac^*} k_b^{*c} + \frac{1}{2} \partial_a \tau_{bc} D^c) \psi^a \lambda^b. \tag{2.59}$$

We rewrite it as

$$\frac{1}{\sqrt{2}} \left( g_{ac^*} k_b^{*c} - \frac{1}{4} \partial_a \tau_{bc} g^{cd} \mathfrak{D}_d \right) \psi^a \lambda^b + \frac{1}{2\sqrt{2}} \partial_a \tau_{bc}(D^c + \frac{1}{2} g^{cd} \mathfrak{D}_d) \psi^a \lambda^b. \tag{2.60}$$

The invariance of the first term is guaranteed by (2.48), and thus we find

$$\mathfrak{R} : D^c + \frac{1}{2} g^{cd} \mathfrak{D}_d \to -(D^c + \frac{1}{2} g^{cd} \mathfrak{D}_d) \tag{2.61}$$

for the invariance. Lastly, let us turn to the bosonic terms including $D^a$

$$\frac{1}{2}(\tau_2)_{ab} D^a D^b + \frac{1}{2} D^a \left( \mathfrak{D}_a + 2\sqrt{2}\xi \delta_a^0 \right). \tag{2.62}$$

We rewrite it as

$$\frac{1}{2} g_{ab} \left( D^a + \frac{1}{2} g^{ac}(\mathfrak{D}_c + 2\sqrt{2}\xi \delta_c^0) \right) \left( D^b + \frac{1}{2} g^{bd}(\mathfrak{D}_d + 2\sqrt{2}\xi \delta_d^0) \right)$$

$$- \frac{1}{8} g^{ab} \left( \mathfrak{D}_a + 2\sqrt{2}\xi \delta_a^0 \right) \left( \mathfrak{D}_b + 2\sqrt{2}\xi \delta_b^0 \right). \tag{2.63}$$

11

The first term in (2.63) is obviously invariant under the action (2.49) and (2.61). The last term is also invariant under the action (2.49) because the term linear in $\xi$ vanishes as is shown in (2.53).

As a result, we have found that the off-shell action $\mathcal{L}$ (2.26) is invariant under the discrete $\mathfrak{R}$-symmetry (2.39), (2.49), (2.56) and (2.61) if we choose $\tau_{ab}$ as (2.44) and $W$ as (2.47). For completeness, we present the off-shell action of our $U(N)$ gauge model which is invariant under the discrete $\mathfrak{R}$-symmetry;

$$
\begin{aligned}
\mathcal{L} \;=\; & -g_{ab^*}\mathcal{D}_m A^a \mathcal{D}^m A^{*b} - \frac{1}{4}g_{ab}v^a_{mn}v^{bmn} - \frac{1}{8}\mathrm{Re}(\mathcal{F}_{ab})\epsilon^{mnpq}v^a_{mn}v^b_{pq} \\
& -\frac{1}{2}\mathcal{F}_{ab}\lambda^a\sigma^m\mathcal{D}_m\bar{\lambda}^b - \frac{1}{2}\mathcal{F}^*_{ab}\mathcal{D}_m\lambda^a\sigma^m\bar{\lambda}^b - \frac{1}{2}\mathcal{F}_{ab}\psi^a\sigma^m\mathcal{D}_m\bar{\psi}^b - \frac{1}{2}\mathcal{F}^*_{ab}\mathcal{D}_m\psi^a\sigma^m\bar{\psi}^b \\
& +g_{ab^*}F^a F^{*b} + F^a\partial_a W + F^{*a}\partial_{a^*}W^* + \frac{1}{2}g_{ab}D^a D^b + \frac{1}{2}D^a\left(\mathfrak{D}_a + 2\sqrt{2}\xi\delta^0_a\right) \\
& +(\frac{i}{4}\mathcal{F}_{abc}F^{*c} - \frac{1}{2}\partial_a\partial_b W)\psi^a\psi^b + \frac{i}{4}\mathcal{F}_{abc}F^c\lambda^a\lambda^b + \frac{1}{\sqrt{2}}(g_{ac^*}k^{*c}_b + \frac{1}{2}\mathcal{F}_{abc}D^c)\psi^a\lambda^b \\
& +(-\frac{i}{4}\mathcal{F}^*_{abc}F^c - \frac{1}{2}\partial_{a^*}\partial_{b^*}W^*)\bar{\psi}^a\bar{\psi}^b - \frac{i}{4}\mathcal{F}^*_{abc}F^{*c}\bar{\lambda}^a\bar{\lambda}^b + \frac{1}{\sqrt{2}}(g_{ca^*}k_b{}^c + \frac{1}{2}\mathcal{F}^*_{abc}D^c)\bar{\psi}^a\bar{\lambda}^b \\
& -i\frac{\sqrt{2}}{8}(\mathcal{F}_{abc}\psi^c\sigma^n\bar{\sigma}^m\lambda^a - \mathcal{F}^*_{abc}\bar{\lambda}^a\bar{\sigma}^m\sigma^n\bar{\psi}^c)v^b_{mn} \\
& -\frac{i}{8}\mathcal{F}_{abcd}\psi^c\psi^d\lambda^a\lambda^b + \frac{i}{8}\mathcal{F}^*_{abcd}\bar{\psi}^c\bar{\psi}^d\bar{\lambda}^a\bar{\lambda}^b,
\end{aligned}
\tag{2.64}
$$

where $g_{ab^*} = \mathrm{Im}(\mathcal{F}_{ab})$ and $W = eA^0 + m\mathcal{F}_0$. In the above expression, the covariant derivatives are defined as

$$
\mathcal{D}_m\Psi^a \;=\; \partial_m\Psi^a - \frac{1}{2}f^a_{bc}v^b_m\Psi^c, \qquad \Psi^a = \{A^a, \psi^a, \lambda^a\},
\tag{2.65}
$$

$$
v^a_{mn} \;=\; \partial_m v^a_n - \partial_n v^a_m - \frac{1}{2}f^a_{bc}v^b_m v^c_n.
\tag{2.66}
$$

By the reasoning we explained at the beginning of this section, our action (2.33) and (2.64) are invariant under $\mathcal{N} = 2$ supersymmetry.

## III.  Extended Supersymmetry Transformation

Our action is manifestly invariant under the $\mathcal{N} = 1$ supersymmetry transformation. We have made our action invariant under the discrete transformation $\mathfrak{R}$, and the algebra of extended supersymmetry permits us to argue for the invariance of our action under the extended $\mathcal{N} = 2$ supersymmetry transformation. In this section, we will first lift the $\mathcal{N} = 1$

supersymmetry transformation

$$\delta_{\eta_1} A^a = \sqrt{2}\eta_1\psi^a, \tag{3.1}$$

$$\delta_{\eta_1}\psi^a = i\sqrt{2}\sigma^m\bar{\eta}_1\mathcal{D}_m A^a + \sqrt{2}\eta_1 F^a, \tag{3.2}$$

$$\delta_{\eta_1} v_m^a = i\eta_1\sigma_m\bar{\lambda}^a - i\lambda^a\sigma_m\bar{\eta}_1, \tag{3.3}$$

$$\delta_{\eta_1}\lambda^a = \sigma^{mn}\eta_1 v_{mn}^a + i\eta_1 D^a, \tag{3.4}$$

to its $\mathcal{N} = 2$ counterpart by exploiting the discrete symmetry $\mathfrak{R}$. We will subsequently examine $SU(2)$ covariance of the $\mathcal{N} = 2$ supersymmetry transformation obtained.

Let us first form a following doublet of fermions;

$$\boldsymbol{\lambda}_i{}^a \equiv \begin{pmatrix} \lambda^a \\ \psi^a \end{pmatrix}, \qquad \boldsymbol{\lambda}^{ia} \equiv \epsilon^{ij}\boldsymbol{\lambda}_j{}^a = \begin{pmatrix} +\psi^a \\ -\lambda^a \end{pmatrix}, \tag{3.5}$$

$$\bar{\boldsymbol{\lambda}}^{ia} \equiv \overline{\boldsymbol{\lambda}_i{}^a} = \begin{pmatrix} \bar{\lambda}^a \\ \bar{\psi}^a \end{pmatrix}, \qquad \bar{\boldsymbol{\lambda}}_i{}^a \equiv \epsilon_{ik}\bar{\boldsymbol{\lambda}}^{ka} = \begin{pmatrix} -\bar{\psi}^a \\ +\bar{\lambda}^a \end{pmatrix} = -\overline{\boldsymbol{\lambda}^{ia}}. \tag{3.6}$$

We carry out the raising and the lowering of $i, j$ indices by $\epsilon^{ij}$ ; $\epsilon^{12} = \epsilon_{21} = 1$, $\epsilon^{21} = \epsilon_{12} = -1$. Recall the action of $\mathfrak{R}$;

$$\mathfrak{R} : \boldsymbol{\lambda}_i{}^a = \begin{pmatrix} +\lambda^a \\ +\psi^a \end{pmatrix} \longrightarrow \boldsymbol{\lambda}^{ia} = \begin{pmatrix} +\psi^a \\ -\lambda^a \end{pmatrix},$$

$$\bar{\boldsymbol{\lambda}}_i{}^a = \begin{pmatrix} -\bar{\psi}^a \\ +\bar{\lambda}^a \end{pmatrix} \longrightarrow \bar{\boldsymbol{\lambda}}^{ia} = \begin{pmatrix} +\bar{\lambda}^a \\ +\bar{\psi}^a \end{pmatrix}, \tag{3.7}$$

and therefore the terms $\hat{F}^a$ in (2.31) and $\hat{D}^a$ in (2.30) which are bilinear in fermions undergo the action;

$$\mathfrak{R} : \begin{array}{c} \hat{F}^a \longrightarrow \hat{F}^{*a} \\ \hat{D}^a \longrightarrow -\hat{D}^a \end{array}. \tag{3.8}$$

Note that this is nothing but (2.61), (2.56). The bosonic fields $A^a$, $v_m^a$ are invariant under $\mathfrak{R}$. So from (3.1), (3.3), we see that the grassman parameter $\eta_2$ for the second supersymmetry forms a doublet with $\eta_1$ such that

$$\mathfrak{R} : \boldsymbol{\eta}_i \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \longrightarrow \begin{pmatrix} +\eta_2 \\ -\eta_1 \end{pmatrix} \equiv \boldsymbol{\eta}^i \equiv \epsilon^{ij}\boldsymbol{\eta}_j \ . \tag{3.9}$$

Demanding the covariance under $\mathfrak{R}$, we obtain the extended supersymmetry transforma-

tion;

$$\boldsymbol{\delta} A^a = \sqrt{2}\boldsymbol{\eta}_j\boldsymbol{\lambda}^{ja}, \tag{3.10}$$

$$\boldsymbol{\delta\lambda}_j{}^a = \sigma^{mn}\boldsymbol{\eta}_j v^a_{mn} + \sqrt{2}i\left(\sigma^m\bar{\boldsymbol{\eta}}_j\right)\mathcal{D}_m A^a + i\left(\begin{array}{cc} \hat{D}^a & +i\sqrt{2}\hat{F}^{*a} \\ -i\sqrt{2}\hat{F}^a & -\hat{D}^a \end{array}\right)\left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right) - \frac{i}{2}\boldsymbol{\eta}_j g^{ab}\mathfrak{D}_b$$

$$-\sqrt{2}ig^{ab^*}\frac{\partial}{\partial A^{*b}}\left(\begin{array}{cc} \xi A^{*0} & +i(eA^{*0}+m\mathcal{F}_0^*) \\ -i(eA^{*0}+m\mathcal{F}_0^*) & -\xi A^{*0} \end{array}\right)\left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right), \tag{3.11}$$

$$\boldsymbol{\delta} v^a_m = i\boldsymbol{\eta}_j\sigma_m\bar{\boldsymbol{\lambda}}^{ja} - i\boldsymbol{\lambda}_j{}^a\sigma_m\bar{\boldsymbol{\eta}}^j . \tag{3.12}$$

Here

$$\bar{\boldsymbol{\eta}}^j \equiv \left(\begin{array}{c} \bar{\eta}_1 \\ \bar{\eta}_2 \end{array}\right) \quad \text{and} \quad \bar{\boldsymbol{\eta}}_j \equiv \epsilon_{ji}\bar{\boldsymbol{\eta}}^i = \left(\begin{array}{c} -\bar{\eta}_2 \\ +\bar{\eta}_1 \end{array}\right). \tag{3.13}$$

The transformation (3.11) is further recast into the following form;

$$\boldsymbol{\delta\lambda}_j{}^a = (\sigma^{mn}\boldsymbol{\eta}_j)v^a_{mn} + \sqrt{2}i(\sigma^m\bar{\boldsymbol{\eta}}_j)\mathcal{D}_m A^a + i(\boldsymbol{\tau}\cdot\boldsymbol{D}^a)_j{}^k\boldsymbol{\eta}_k - \frac{1}{2}\boldsymbol{\eta}_j f^a{}_{bc}A^{*b}A^c, \tag{3.14}$$

$$\boldsymbol{\delta\bar{\lambda}}^{ja} = -(\bar{\boldsymbol{\eta}}^j\bar{\sigma}^{mn})v^a_{mn} - \sqrt{2}i(-\boldsymbol{\eta}^j\sigma^m)\mathcal{D}_m A^{*a} - i\bar{\boldsymbol{\eta}}^k(\boldsymbol{\tau}\cdot\boldsymbol{D}^{*a})_k{}^j - \frac{1}{2}\bar{\boldsymbol{\eta}}^j f^a{}_{bc}A^b A^{*c}, \tag{3.15}$$

$$\boldsymbol{D}^a = \hat{\boldsymbol{D}}^a - \sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(\boldsymbol{\mathcal{E}}A^{*0} + \boldsymbol{\mathcal{M}}\mathcal{F}_0^*\right). \tag{3.16}$$

Here

$$\hat{\boldsymbol{D}}^a = (\hat{D}_1^a, \hat{D}_2^a, \hat{D}_3^a), \quad \left\{\begin{array}{rcl} \hat{D}_1^a + i\hat{D}_2^a & = & -i\sqrt{2}\hat{F}^a, \\ \hat{D}_1^a - i\hat{D}_2^a & = & +i\sqrt{2}\hat{F}^{*a}, \\ \hat{D}_3^a & = & \hat{D}^a, \end{array}\right. \tag{3.17}$$

$$\boldsymbol{\mathcal{E}} = (0, -e, \xi), \tag{3.18}$$

$$\boldsymbol{\mathcal{M}} = (0, -m, 0), \tag{3.19}$$

and $\boldsymbol{\tau}$ are the Pauli matrices. We have used (2.52) in the last term of (3.14) and that of (3.15). Finally, we can easily check that (2.2) in fact holds in these transformations.

Let us now examine the $SU(2)$ covariance of the extended susy transformation given by (3.10), (3.11), (3.12). All except the last term in (3.11) are manifestly covariant under the rigid $SU(2)$ transformations. In particular, $\hat{\boldsymbol{D}}^a$, given by (2.30)$\sim$(2.32) which are bilinear in fermions, transforms as a real triplet under $SU(2)$,

$$i\boldsymbol{\tau}\cdot\hat{\boldsymbol{D}}^a = i\sqrt{2}\left(\begin{array}{cc} \hat{D}^a & i\sqrt{2}\hat{F}^{*a} \\ -i\sqrt{2}\hat{F}^a & -\hat{D}^a \end{array}\right) = g^{ab^*}g_{cb^*,d}\boldsymbol{\lambda}_j^{\{c}\boldsymbol{\lambda}^{d\}k} + g^{ab^*}g_{cb^*,d^*}\bar{\boldsymbol{\lambda}}_j^{\{c}\bar{\boldsymbol{\lambda}}^{d\}k}. \tag{3.20}$$

The last term in (3.11) is $SU(2)$ covariant provided the two three-dimensional real vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{M}}$ transform as triplets. Their actual form (3.18) and (3.19) tell us that the rigid $SU(2)$ has been gauge fixed in this six-dimensional parameter space of $(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{M}})$, by making these two vectors point to a specific direction. The manifest $SU(2)$ covariance is lost at this point. The transformation law we have exhibited generalizes the one seen in the literature [3] by the inclusion of the $\xi$ term and the superpotential.

A very important property of the triplet of the auxiliary fields $\boldsymbol{D}^a$ is that it is complex as opposed to be real. Indeed, it has a constant imaginary part;

$$\text{Im } \boldsymbol{D}^a = \delta^a{}_0(-\sqrt{2}m)\,(0,1,0)\,. \tag{3.21}$$

This supplies an essential ingredient for partial breaking of $\mathcal{N} = 2$ supersymmetry in the next section.

The supersymmetry transformation law for the auxiliary fields is determined by requiring the closure of the $\eta_1$- and $\eta_2$-supersymmetries;

$$
\begin{aligned}
\delta F^a &= -i\sqrt{2}\mathcal{D}_m\psi^a\sigma^m\bar{\eta}_1 - \bar{\eta}_1\bar{\lambda}^b k_b{}^a \\
&\quad + \delta_{\eta_2}(g^{ab}\partial_b W - g^{ab}\partial_{b^*}W^*) + i\sqrt{2}\eta_2\sigma^m\mathcal{D}_m\bar{\lambda}^a + \eta_2\psi^b k_b^{*a}, \tag{3.22} \\
\delta D^a &= -\eta_1\sigma^m\mathcal{D}_m\bar{\lambda}^a - \mathcal{D}_m\lambda^a\sigma^m\bar{\eta}_1 \\
&\quad - \delta_{\eta_2}(g^{ab}\mathfrak{D}_b) - \eta_2\sigma^m\mathcal{D}_m\bar{\psi}^a - \mathcal{D}_m\psi^a\sigma^m\bar{\eta}_2, \tag{3.23}
\end{aligned}
$$

where the $\mathcal{D}_m$ represents the gauge covariant derivative (2.65). The supersymmetry transformation forms the algebra

$$[\delta_\eta, \delta_{\eta'}]\Psi^a = -2i(\eta\sigma^m\bar{\eta}' - \eta'\sigma^m\bar{\eta})\mathcal{D}_m\Psi^a, \quad \Psi^a = \{A^a, \psi^a, F^a, v^a_{mn}, \lambda^a, D^a\} \tag{3.24}$$

where $(\eta, \eta') = (\eta_1, \eta_1')$ or $(\eta_2, \eta_2')$.

## IV.   Some Properties of the vacuum

In order to discuss properties of our model, let us fix the form of $\mathcal{F}$. The first equation in (2.12) implies that $k_a{}^b = f^b_{ac}A^c$ and thus $k_0{}^a = k_a{}^0 = 0$, while the second equation in (2.12) implies that

$$k_{\hat{a}}{}^{\hat{b}}\partial_{\hat{b}}\mathcal{F}_0 = 0, \tag{4.1}$$

as well as $k_{\hat{a}}{}^{\hat{b}}\partial_{\hat{b}}\mathcal{F}_{\hat{c}} = -f^{\hat{b}}_{\hat{a}\hat{c}}\mathcal{F}_{\hat{b}}$. An obvious solution to (4.1) is

$$\mathcal{F} = f(A^0) + cA^0\,\mathcal{G}(\hat{B}) + \hat{\mathcal{F}}(\hat{A}), \tag{4.2}$$

15

where $f(A^0)$, $\mathcal{G}(\hat{B})$ and $\hat{\mathcal{F}}(\hat{A})$ are analytic functions of $A^0$, $\hat{B} = \mathrm{Tr}(\hat{A}^2)/2c_2$ and a trace function of $\hat{A} = A^{\hat{a}}t_{\hat{a}}$, respectively. We can choose $\mathcal{G}(0) = 0$ without loss of generality. The constant $c_2$ is the quadratic Casimir defined by $\mathrm{Tr}(t_{\hat{a}}t_{\hat{b}}) = c_2\delta_{\hat{a}\hat{b}}$. One finds that for this prepotential the Kähler metric becomes

$$g_{00^*} = \mathrm{Im}(f_{00}), \quad g_{\hat{a}0^*} = \delta_{\hat{a}\hat{b}}\mathrm{Im}(\mathcal{G}'cA^{\hat{b}}), \quad g_{\hat{a}\hat{b}^*} = \mathrm{Im}\left(cA^0(\delta_{\hat{a}\hat{b}}\mathcal{G}' + \mathcal{G}''\delta_{\hat{a}\hat{c}}\delta_{\hat{b}\hat{d}}A^{\hat{c}}A^{\hat{d}}) + \hat{\mathcal{F}}_{\hat{a}\hat{b}}\right). \quad (4.3)$$

Note that the $U(1)$ part and the $SU(N)$ part have non-trivial mixings as long as $c \neq 0$. In the following we examine the model specified by (4.2).

Let us first examine the local minimum of the scalar potential $\mathcal{V} \equiv -\mathcal{L}_{\mathrm{pot}}$

$$\begin{aligned}
\mathcal{V} &= g^{ab}\left(\frac{1}{8}\mathfrak{D}_a\mathfrak{D}_b + \xi^2\delta_a^0\delta_b^0 + \partial_a W\partial_{b^*}W^*\right) \\
&= g^{ab}\left(\frac{1}{8}\mathfrak{D}_a\mathfrak{D}_b + \partial_a\left(\mathcal{E}A^0 + \mathcal{M}\mathcal{F}_0\right)\cdot\partial_{b^*}\left(\mathcal{E}A^0 + \mathcal{M}\mathcal{F}_0\right)^*\right),
\end{aligned} \quad (4.4)$$

where we have used (2.53). Here, we consider the unbroken $SU(N)$ phase at which the $A^{\hat{a}}$ do not acquire vacuum expectation values. Substituting $A^{\hat{a}} = 0$ into the equation

$$\begin{aligned}
0 = \delta\mathcal{V}/\delta A^a &= -g^{bd}\partial_a g_{de}g^{ec}\left(\frac{1}{8}\mathfrak{D}_b\mathfrak{D}_c + \xi^2\delta_b^0\delta_c^0 + \partial_b W\partial_{c^*}W^*\right) \\
&\quad + g^{bc}\left(\frac{1}{4}\mathfrak{D}_b\partial_a\mathfrak{D}_c + \partial_a\partial_b W\partial_{c^*}W^*\right),
\end{aligned} \quad (4.5)$$

we obtain

$$\frac{i}{2}f_{000}g_{00}^{-2}\delta_a^0\left(\xi^2 + (e + mf_{00})(e + mf_{00}^*)\right) + g_{00}^{-1}mf_{000}\delta_a^0(e + mf_{00}^*) = 0. \quad (4.6)$$

Here we have derived

$$\langle\mathfrak{D}_a\rangle = 0, \quad \langle\partial_a W\rangle = \delta_a^0(e + mf_{00}), \quad \partial_0\partial_a W = \delta_a^0 mf_{000}, \quad \partial_a g_{00} = -\frac{i}{2}f_{000}\delta_a^0, \quad (4.7)$$

as well as

$$\langle g^{00}\rangle = g_{00}^{-1}, \quad \langle g^{0\hat{a}}\rangle = 0. \quad (4.8)$$

The expressions with bracket $\langle\cdots\rangle$ imply $\cdots$ evaluated at $A^{\hat{a}} = 0$. It is obvious that (4.6) is satisfied when $f_{000} = 0$, but it is a saddle point because $\langle\partial_0\partial_{0^*}\mathcal{V}\rangle = 0$, and thus does not represent a stable vacuum. The stable minimum is at

$$f_{00} = -\frac{e}{m} \pm i\frac{\xi}{m}. \quad (4.9)$$

We shall show that at the stable minimum (4.9) massless fermions emerge. For this purpose, we examine the fermion mass term

$$\begin{aligned}
\mathcal{L}_{\mathrm{mass}} &= -\frac{i}{4}g^{cd^*}\partial_c\tau_{ab}\partial_{d^*}W^*(\psi^a\psi^b + \lambda^a\lambda^b) \\
&\quad + \frac{1}{2\sqrt{2}}\left(g_{ac^*}k_b^{*c} - g_{bc^*}k_a^{*c} - \sqrt{2}\xi\delta_c^0(\tau_2^{-1})^{cd}\partial_a\tau_{bd}\right)\psi^a\lambda^b + c.c. \quad (4.10)
\end{aligned}$$

Substituting $A^{\hat{a}} = 0$ into this mass term $\mathcal{L}_{\text{mass}}$, we find that the the $U(1)$ fermions and the $SU(N)$ fermions decouple because $\langle \mathcal{F}_{00\hat{a}} \rangle = 0$,

$$
\mathcal{L}_{\text{mass}} = \frac{1}{2}\boldsymbol{\lambda}_i{}^0 M^{ij}_{U(1)} \boldsymbol{\lambda}_j{}^0 + \frac{1}{2}\delta_{\hat{a}\hat{b}} \boldsymbol{\lambda}_i{}^{\hat{a}} M^{ij}_{SU(N)} \boldsymbol{\lambda}_j{}^{\hat{b}} + c.c.,
$$

$$
M^{ij}_{U(1)} = -\frac{i}{2}g_{00}^{-1} f_{000} \begin{pmatrix} e + m f_{00}^* & -i\xi \\ -i\xi & e + m f_{00}^* \end{pmatrix},
$$

$$
M^{ij}_{SU(N)} = -\frac{i}{2}g_{00}^{-1} c \langle \mathcal{G}' \rangle \begin{pmatrix} e + m f_{00}^* & -i\xi \\ -i\xi & e + m f_{00}^* \end{pmatrix}. \tag{4.11}
$$

It is easy to diagonalize these mass matrices and one finds that the $U(1)$ fermions $\frac{1}{\sqrt{2}}(\lambda^0 \pm \psi^0)$ acquire masses $\left| -\frac{i}{2}g_{00}^{-1} f_{000}(e + m f_{00}^* \mp i\xi) \right|$, while the $SU(N)$ fermions $\frac{1}{\sqrt{2}}(\lambda^{\hat{a}} \pm \psi^{\hat{a}})$ acquire masses $\left| -\frac{i}{2}g_{00}^{-1} c \langle \mathcal{G}' \rangle (e + m f_{00}^* \mp i\xi) \right|$.

At the stable minimum $f_{00} = -\frac{e}{m} \pm i\frac{\xi}{m}$, the $U(1)$ fermion $\frac{1}{\sqrt{2}}(\lambda^0 \mp \psi^0)$ and the $SU(N)$ fermions $\frac{1}{\sqrt{2}}(\lambda^{\hat{a}} \mp \psi^{\hat{a}})$ remain massless, while the $U(1)$ fermion $\frac{1}{\sqrt{2}}(\lambda^0 \pm \psi^0)$ and the $SU(N)$ fermions $\frac{1}{\sqrt{2}}(\lambda^{\hat{a}} \pm \psi^{\hat{a}})$ become massive with masses, $|-m \langle\!\langle f_{000} \rangle\!\rangle|$ and $|-mc \langle\!\langle \mathcal{G}' \rangle\!\rangle|$, respectively. Here, $\langle\!\langle \cdots \rangle\!\rangle$ is the expectation value of $\cdots$ at the vacuum. The $U(1)$ massless fermion is regarded as the Nambu-Goldstone fermion.

Let us demonstrate this last statement from the transformation law (3.11). Taking the expectation value, we see

$$
\langle\!\langle \boldsymbol{\delta\lambda}^0 \rangle\!\rangle = -\sqrt{2}i \langle\!\langle g^{00} \rangle\!\rangle \begin{pmatrix} \xi & i \langle\!\langle e + m f_{00}^* \rangle\!\rangle \\ -i \langle\!\langle e + m f_{00}^* \rangle\!\rangle & -\xi \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}
$$

$$
= \mp\sqrt{2}im \begin{pmatrix} 1 & \pm 1 \\ \mp 1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{4.12}
$$

$$
\langle\!\langle \boldsymbol{\delta\lambda}^{\hat{a}} \rangle\!\rangle = 0. \tag{4.13}
$$

We have used (4.8), (4.9). Therefore,

$$
\left\langle\!\!\left\langle \frac{\delta(\lambda^0 \mp \psi^0)}{\sqrt{2}} \right\rangle\!\!\right\rangle = \mp 2mi(\eta_1 \pm \eta_2),
$$

$$
\left\langle\!\!\left\langle \frac{\delta(\lambda^0 \pm \psi^0)}{\sqrt{2}} \right\rangle\!\!\right\rangle = 0. \tag{4.14}
$$

One linear combination of the $U(1)$ fermion, $\frac{1}{\sqrt{2}}(\lambda^0 \mp \psi^0)$, is in fact the Nambu-Goldstone fermion.

Finally, let us discuss a mechanism which is responsible for partial breaking of $\mathcal{N} = 2$ supersymmetry to be realized. We see that partial breaking requires that the $2 \times 2$ matrix

$\langle\!\langle \boldsymbol{\tau} \cdot \boldsymbol{D}^a \rangle\!\rangle$ in (3.14) has one nonvanishing eigenvalue for some $a$. We obtain

$$
\begin{aligned}
- \langle\!\langle \det \boldsymbol{\tau} \cdot \boldsymbol{D}^a \rangle\!\rangle \;&=\; \langle\!\langle \boldsymbol{D}^a \cdot \boldsymbol{D}^a \rangle\!\rangle \\
&=\; \langle\!\langle \operatorname{Re}\boldsymbol{D}^a \cdot \operatorname{Re}\boldsymbol{D}^a \rangle\!\rangle - \langle\!\langle \operatorname{Im}\boldsymbol{D}^a \cdot \operatorname{Im}\boldsymbol{D}^a \rangle\!\rangle + 2i \langle\!\langle \operatorname{Re}\boldsymbol{D}^a \cdot \operatorname{Im}\boldsymbol{D}^a \rangle\!\rangle \\
&=\; 0, \tag{4.15}
\end{aligned}
$$

which implies that partial breaking is certainly not possible without nonvanishing imaginary part of $\boldsymbol{D}^a$. Using (3.21), we convert this condition into

$$
\begin{aligned}
\langle\!\langle \operatorname{Re}\boldsymbol{D}^{\hat a} \rangle\!\rangle \;&=\; 0, \\
\| \langle\!\langle \operatorname{Re}\boldsymbol{D}^0 \rangle\!\rangle \| = \| \operatorname{Im}\boldsymbol{D}^0 \| \;&=\; \sqrt{2}m, \\
\langle\!\langle \operatorname{Re}\boldsymbol{D}^0 \rangle\!\rangle \cdot \operatorname{Im}\boldsymbol{D}^0 \;&=\; 0.
\end{aligned} \tag{4.16}
$$

Coming back to the extremum condition (4.5) of the scalar potential at the unbroken $SU(N)$ phase, we see that it can also be converted as

$$
0 = \frac{\delta\mathcal{V}}{\delta A^a} = \frac{i}{4} \langle\!\langle f_{000}\delta_a^0 \rangle\!\rangle \langle\!\langle \boldsymbol{D}^0 \rangle\!\rangle \cdot \langle\!\langle \boldsymbol{D}^0 \rangle\!\rangle . \tag{4.17}
$$

The condition for a stable vacuum is obviously equivalent to that of partial supersymmetry breaking (4.16). Note that at the vacuum

$$
\langle\!\langle \mathcal{V} \rangle\!\rangle = \langle\!\langle g_{00}^{-1} \rangle\!\rangle \left( \xi^2 + \langle\!\langle |e + mf_{00}|^2 \rangle\!\rangle \right) = \pm 2m\xi \neq 0 . \tag{4.18}
$$

## V.  $\mathcal{N} = 2$ Supercurrents

In the previous section, the rigid $SU(2)$ symmetry, in particular, its discrete element $\mathfrak{R}$ has been exploited to provide $N = 2$ supersymmetry of our model. In this section, we discuss another rigid transformation, namely, the one associated with the $U(1)_R$ transformation and the attendant supermultiplet of currents.

It is well known that the Wess-Zumino model consisting of the scalar superfield with a superpotential permits the $U(1)_R$ current, the supercurrent and the energy momentum tensor as a supermultiplet of currents when the superpotential is a monomial in scalar superfield [36]. It is then possible to assign $R$ weight one to the superpotential. (Extended) supermultiplet of currents exists for $(\mathcal{N} = 2)$ super Yang-Mills as well [36] [33]. Starting from the $U(1)_R$ current, we can use this multiplet structure to derive the form of the supercurrent and the energy momentum tensor and to check the consistency of supersymmetry algebra. We illustrate this in the Wess-Zumino model in the Appendix A. Our model has $N = 2$ supermultiplet of Noether currents when it is possible to assign $R$ weight two to the prepotential $\mathcal{F}$. We show how this is used to derive the $\mathcal{N} = 2$ supercurrents for generic $\mathcal{F}$.

The $R$ transformation is given by

$$R: \quad \Phi(x,\theta,\bar{\theta}) \quad \to \quad e^{i\alpha}\Phi(x,e^{-\frac{i\alpha}{2}}\theta,\bar{\theta}),$$

$$\mathcal{W}_\alpha(x,\theta,\bar{\theta}) \quad \to \quad \mathcal{W}_\alpha(x,e^{-\frac{i\alpha}{2}}\theta,\bar{\theta}), \tag{5.1}$$

$$R: A \quad \to \quad e^{i\alpha}A \quad , \quad v_m \to v_m,$$

$$\psi \quad \to \quad e^{\frac{i\alpha}{2}}\psi \quad , \quad \lambda \to e^{\frac{i\alpha}{2}}\lambda, \tag{5.2}$$

$$F \quad \to \quad F \quad , \quad D \to D.$$

We assume that the prepotential $\mathcal{F}$ is transformed as weight two under $R$

$$\mathcal{F} \to e^{2i\alpha}\mathcal{F}. \tag{5.3}$$

The $U(1)_R$ current associated is

$$\theta J\bar{\theta} \quad \equiv \quad (\tau_2)_{ab}\left(\theta\bar{\boldsymbol{\lambda}}^{ia}\boldsymbol{\lambda}_i^{\ b}\theta + iA^{*a}\theta\overleftrightarrow{\mathcal{D}}\cdot\sigma\bar{\theta}A^b\right) \tag{5.4}$$

$$\equiv \quad (\tau_2)_{ab}\left(\theta j^{ab}\bar{\theta} + 2\theta\Delta j^{ab}\bar{\theta}\right) . \tag{5.5}$$

The second term is known as the improvement term. Using the transformation law of rigid $\mathcal{N}=2$ supersymmetry in section III, we obtain

$$\theta\boldsymbol{\delta}J\bar{\theta} = (\tau_2)_{ab}\left(\theta\boldsymbol{\delta}j^{ab}\bar{\theta} + 2\theta\boldsymbol{\delta}(\Delta j)^{ab}\bar{\theta}\right) + \boldsymbol{\delta}(\tau_2)_{ab}\left(\theta j^{ab}\bar{\theta} + 2\theta(\Delta j)^{ab}\bar{\theta}\right), \tag{5.6}$$

where

$$\theta\boldsymbol{\delta}j^{ab}\bar{\theta}$$

$$= \bar{\theta}\bar{\boldsymbol{\lambda}}^{jb}\left((\theta\sigma^{mn}\boldsymbol{\eta}_j)v_{mn}^{\ a} + \sqrt{2}i(\theta\sigma^m\bar{\boldsymbol{\eta}}_j)\mathcal{D}_mA^a + i(\boldsymbol{\tau}\cdot\boldsymbol{D}^a)_j^{\ k}(\theta\boldsymbol{\eta}_k) - \frac{1}{2}(\theta\boldsymbol{\eta}_j)f^a_{\ cd}A^{*c}A^d\right)$$

$$- \left((\bar{\boldsymbol{\eta}}^j\bar{\sigma}^{mn}\bar{\theta})v_{mn}^{\ b} + \sqrt{2}i(-\boldsymbol{\eta}^j\sigma^m\bar{\theta})\mathcal{D}_mA^{*b} + i(\bar{\boldsymbol{\eta}}^k\bar{\theta})(\boldsymbol{\tau}\cdot\boldsymbol{D}^{*b})_k^{\ j} + \frac{1}{2}(\bar{\boldsymbol{\eta}}^j\bar{\theta})f^b_{\ ef}A^eA^{*f}\right)\theta\lambda_j^{\ a}, \tag{5.7}$$

$$\theta\boldsymbol{\delta}(\Delta j^{ab})\bar{\theta} = \frac{\sqrt{2}}{2}iA^{*a}\theta\overleftrightarrow{\mathcal{D}}_m\sigma^m\bar{\theta}\boldsymbol{\eta}_j\boldsymbol{\lambda}^{jb} + \frac{\sqrt{2}}{2}i\bar{\boldsymbol{\eta}}^j\bar{\boldsymbol{\lambda}}_j^{\ a}\theta\overleftrightarrow{\mathcal{D}}_m\sigma^m\bar{\theta}A^b + \frac{i}{2}A^{*a}\theta\boldsymbol{\delta}\overleftrightarrow{\mathcal{D}}_m\sigma^m\theta A^b, \tag{5.8}$$

$$2i\boldsymbol{\delta}(\tau_2)_{ab} = \sqrt{2}\left(\tau_{abc}(A^d)\boldsymbol{\eta}_i\boldsymbol{\lambda}^{ic} - \tau^*_{abc}(A^{*d})\bar{\boldsymbol{\eta}}^i\bar{\boldsymbol{\lambda}}_i^{\ c}\right). \tag{5.9}$$

In the case where the prepotential is a degree two polynomial in $A^a$, $\boldsymbol{\delta}(\tau_2)_{ab} = 0$ and eq.(5.6) provides construction of $\mathcal{N}=2$ improved supercurrents which are conserved;

$$\boldsymbol{\eta}_j\boldsymbol{\mathcal{S}}^{(j)m} + \bar{\boldsymbol{\eta}}^j\bar{\boldsymbol{S}}_{(j)}^{\ m} \equiv -\frac{1}{2}(\tau_2)_{ab}\,\mathrm{tr}\,\bar{\sigma}^m\left(\boldsymbol{\delta}(j^{ab}) + 2\boldsymbol{\delta}(\Delta j^{ab})\right). \tag{5.10}$$

19

Here " tr " implies a trace in the spinor space.

The $R$ current is not conserved when $\mathcal{F}$ is not a degree two polynomial in $A$ and the above construction would appear not useful for the general construction of the conserved supercurrents. We will show below that this is not the case. Let us write the prepotential $\mathcal{F}$ generically as

$$\mathcal{F} = \sum_{n,j} h_j^{(n)} C_j^{(n)}(A^a). \tag{5.11}$$

Here $C_j^{(n)}(A^a)$ are $n$-th order $U(N)$ invariant polynomials in $A^a$ properly normalized and labelled by the index $j$, and $h_j^{(n)}$ are their coefficients. We first observe that we can assign weight two to $\mathcal{F}$ in (5.11) provided $h_j^{(n)}$ transform as weight $-(n-2)$. Let us consider the local version of the $U(1)_R$ transformation (5.2), replacing $\alpha$ by $\alpha(x)$. We obtain

$$S[Ae^{i\alpha(x)}, \ \boldsymbol{\lambda}_j e^{\frac{i\alpha(x)}{2}}, \ ...] - S[A, \ \boldsymbol{\lambda}_j, \ ...]$$

$$= \int d^4x \ \partial_m \left( \alpha(x) \left( -\frac{1}{2} \right) \operatorname{tr} \bar{\sigma}^m J \right) + \int d^4x \ \alpha(x) \partial_m \left( \frac{1}{2} \operatorname{tr} \bar{\sigma}^m J \right)$$

$$+ \int d^4x \ i\alpha(x) \sum_{n,j} (n-2) \frac{\partial}{\partial h_j^{(n)}} \mathcal{L}. \tag{5.12}$$

Here $\mathcal{L}$ and $S$ are the Lagrangian and the action of our model respectively. The left hand side vanishes by the equation of motion, and we obtain

$$\partial_m \left( -\frac{1}{2} \operatorname{tr} \bar{\sigma}^m J \right) = i \left( \sum_{n,j} (n-2) \frac{\partial}{\partial h_j^{(n)}} \right) \mathcal{L} \equiv \Delta_h \mathcal{L}. \tag{5.13}$$

Taking the supersymmetry variation of this equation, we obtain

$$\partial_m \left( -\frac{1}{2} \operatorname{tr} \bar{\sigma}^m \boldsymbol{\delta} J \right) = \Delta_h \boldsymbol{\delta} \mathcal{L}. \tag{5.14}$$

As our action is $\mathcal{N} = 2$ supersymmetric, the right hand side is written as

$$\Delta_h \partial_m X^m = \partial_m \Delta_h X^m, \tag{5.15}$$

$$X^m = \boldsymbol{\eta}_j \boldsymbol{y}^j + \bar{\boldsymbol{\eta}}^j \bar{\boldsymbol{y}}_j, \tag{5.16}$$

for some operator $X^m$ linear in $\boldsymbol{\eta}_i$ and $\bar{\boldsymbol{\eta}}^i$. Hence

$$\partial_m \left( -\frac{1}{2} \operatorname{tr} \bar{\sigma}^m \boldsymbol{\delta} J - \Delta_h X^m \right) = 0. \tag{5.17}$$

This provides a general construction of the conserved $\mathcal{N} = 2$ supercurrents of our model;

$$\boldsymbol{\eta}_j \boldsymbol{S}^{(j)m} + \bar{\boldsymbol{\eta}}^j \bar{\boldsymbol{S}}_{(j)}^{\ m} \equiv -\frac{1}{2} \operatorname{tr}(\bar{\sigma}^m \boldsymbol{\delta} J) - \Delta_h X^m. \tag{5.18}$$

20

The form of the supercurrents given in eq.(5.18) tells us that our model does not permit a universal coupling to $\mathcal{N} = 2$ supergravity. The piece $-\Delta_h X^m$ is not generic and depends on the functional form of the prepotential $\mathcal{F}(A)$ in $A$. This and the previous analysis in [22, 24] support the point of view that $\mathcal{N} = 2$ supersymmetric gauge models with nontrivial Kähler potential should be viewed as a low energy effective action.

Let us now further transform (5.18)

$$\boldsymbol{\delta} \left( \boldsymbol{\eta}_j \mathcal{S}^{(j)m} + \bar{\boldsymbol{\eta}}^j \bar{\mathcal{S}}_{(j)}{}^m \right) = -\frac{1}{2} \operatorname{tr} \bar{\sigma}^m \boldsymbol{\delta\delta} J - \Delta_h \boldsymbol{\delta} X^m. \tag{5.19}$$

This generates the $\mathcal{N} = 2$ supersymmetry algebra (1.2) quoted in the introduction and at the same time provides its consistency conditions. Let us note that

$$\begin{aligned}
\theta \boldsymbol{\delta\delta} J \bar{\theta} &= (\tau_2)_{ab} \left( \theta \boldsymbol{\delta\delta} j^{ab} \bar{\theta} + 2\theta \boldsymbol{\delta\delta}(\Delta j)^{ab} \bar{\theta} \right) + 2\boldsymbol{\delta}(\tau_2)_{ab} \left( \theta \boldsymbol{\delta} j^{ab} \bar{\theta} + 2\theta \boldsymbol{\delta}(\Delta j)^{ab} \bar{\theta} \right) \\
&\quad + \boldsymbol{\delta\delta}(\tau_2)_{ab} \left( \theta j^{ab} \bar{\theta} + 2\theta (\Delta j)^{ab} \bar{\theta} \right).
\end{aligned} \tag{5.20}$$

Denote by $\delta_{\eta_j}$ ($\delta_{\bar{\eta}^j}$) the transformation in which only $\eta_j (\bar{\eta}^j)$ is kept in $\boldsymbol{\delta}$. The conditions

$$\begin{aligned}
\delta_{\eta_j} \mathcal{S}^{(j)m} &= 0 \\
\delta_{\bar{\eta}^j} \bar{\mathcal{S}}^{(j)m} &= 0
\end{aligned} \qquad \text{with } j \text{ not summed} \tag{5.21}$$

provide

$$-\frac{1}{2} \operatorname{tr} \bar{\sigma}^m \delta_{\eta_j} \delta_{\eta_j} J - \Delta_h \eta_j \delta_{\eta_j} y^j = 0 \qquad \text{with } j \text{ not summed} \tag{5.22}$$

and its complex conjugate. Their actual expressions are quite involved as one sees from (5.20) and the transformation laws (3.10)$\sim$(3.16). We will not discuss eq.(5.22) further in this paper. In the case where $\mathcal{F}$ is degree two in $A$, $y^j = 0$, and $\boldsymbol{\delta}(\tau_2)_{ab} = 0$, eq.(5.22) can be checked easily as in [33] and in Appendix (B.18) with the aid of the equations of motion.

Let us finally read off the constant matrix $C_i{}^j$ in (1.2) from our algebra (5.19). The only piece in (5.20) which can contribute to $C_i{}^j$ is the part in $(\tau_2)_{ab} \boldsymbol{\delta\delta} j^{ab}$ which is linear both in $\boldsymbol{D}^a$ and in $\boldsymbol{D}^{*a}$. This part is computed as

$$2 (\tau_2)_{ab} \boldsymbol{D}^{*b} \cdot \boldsymbol{D}^a \bar{\theta} (\bar{\boldsymbol{\eta}} \boldsymbol{\eta}) \theta + 2i (\tau_2)_{ab} \left( \boldsymbol{D}^{*b} \times \boldsymbol{D}^a \right) \cdot \bar{\theta} \bar{\boldsymbol{\eta}} \boldsymbol{\tau} \boldsymbol{\eta} \theta. \tag{5.23}$$

Substituting the expressions (3.16) $\sim$ (3.19) into this equation, we find that the second term contains $8m\xi \bar{\theta} \bar{\boldsymbol{\eta}} \tau_1 \boldsymbol{\eta} \theta$. Translated into (1.2), this implies

$$C_i{}^j = +2m\xi (\boldsymbol{\tau}_1)_i{}^j. \tag{5.24}$$

This is consistent with (4.18).

## VI.  Fermionic Shift Symmetry

Equations (4.12) and (4.13) express the extended supersymmetry transformation of the $SU(2)$ doublet of $U(N)$ fermions on the vacuum as $U(1)$ fermionic shift generated by

$$\boldsymbol{\chi}_i \equiv \sqrt{2}m \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{6.1}$$

Note that the coupling constants $e$, $m$, $\xi$ of our model carry dimension two and that $\boldsymbol{\chi}_i$ carry dimension 3/2. The Nambu-Goldstone fermion is the maximal mixing of the $U(1)$ gauge fermion and the $U(1)$ matter fermion.

Restricting our attention to the $U(N)$ field strength gauge superfield $\mathcal{W}_\alpha$, let us recast (4.12) into

$$\langle\!\langle \boldsymbol{\delta}\mathcal{W}_\alpha \rangle\!\rangle = (\mp\chi_1 - \chi_2)\, \mathbf{1}_{N\times N} \equiv 4\pi\chi_\alpha \mathbf{1}_{N\times N}. \tag{6.2}$$

We obtain

$$\langle\!\langle \boldsymbol{\delta}S \rangle\!\rangle = \chi^\alpha \langle\!\langle w_\alpha \rangle\!\rangle, \tag{6.3}$$

$$\langle\!\langle \boldsymbol{\delta}w_\alpha \rangle\!\rangle = N\chi_\alpha, \tag{6.4}$$

where

$$S = \frac{1}{32\pi^2}\,\mathrm{tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha, \quad w_\alpha = \frac{1}{4\pi}\,\mathrm{tr}\,\mathcal{W}_\alpha. \tag{6.5}$$

In this sense, our spontaneously broken supersymmetry is realized on the vacuum as the $U(1)$ fermionic shift noted by ref [37] in the $\mathcal{N} = 2$ $U(N)$ super Yang-Mills deformed by the superpotential $W(\Phi)$. See also [38]. As for its transformation acting on the fields or equivalently on a generic state, let us note that

$$\boldsymbol{\delta}\boldsymbol{\lambda}_j{}^a = \langle\!\langle \boldsymbol{\delta}\boldsymbol{\lambda}_j{}^a \rangle\!\rangle + \cdots. \tag{6.6}$$

Here $\langle\!\langle \boldsymbol{\delta}\boldsymbol{\lambda}_j{}^a \rangle\!\rangle$ is given in (4.12), and $\cdots$ denotes the parts which do not receive the vacuum expectation values. This latter part is to be suppressed by $\frac{1}{m}$ with the replacement $\boldsymbol{\eta}_j \to \frac{\chi_j}{\sqrt{2}m}$ when

$$\frac{e}{m} \ll 1, \quad \frac{\xi}{m} \ll 1, \quad \xi \neq 0 \tag{6.7}$$

for appropriate low energy processes. The spontaneously broken supersymmetry operates as an approximate fermionic $U(1)$ shift symmetry in this regime.

## Acknowledgements

# Appendix    A

In the text, we exploited the extended operation $\mathfrak{R}$ which involves the sign change of the parameter $\xi$ as well as the transformation of the two component spinor parameter $\boldsymbol{\eta}_j$ to demonstrate that our action $\mathcal{L}$ or $\mathcal{L}'$ is invariant under $\mathcal{N} = 2$ supersymmetry. Though the use of $\mathfrak{R}$ is logical from an algebraic point of view, clearly it is not a symmetry in the sense of Noether. In this appendix, we provide another proof, using the more conventional operation which involves the transformation of the fields alone. To be more specific, let $R$ be a generator such that

$$R\lambda^a R^{-1} = \psi^a, \ R\psi^a R^{-1} = -\lambda^a, \ RA^a R^{-1} = A^a, \ \text{and} \ Rv_m{}^a R^{-1} = v_m{}^a. \tag{A.1}$$

Let us start from the eqs. of the $\mathcal{N} = 1$ transformation laws (3.1) $\sim$ (3.4), replacing $\eta$, by $\theta$ and writing $F^a$ and $D^a$ explicitly by (3.16), (3.17).

$$\delta_\theta^{(1,\xi)} A^a = \sqrt{2}\theta\psi^a, \tag{A.2}$$

$$\delta_\theta^{(1,\xi)} \psi^a = i\sqrt{2}\sigma^m\bar{\theta}\mathcal{D}_m A^a + \sqrt{2}\theta\left(\hat{F}^a - \sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(eA^{*0} + m\mathcal{F}_0^*\right)\right), \tag{A.3}$$

$$\delta_\theta^{(1,\xi)} v_m{}^a = i\theta\sigma_m\bar{\lambda}^a - i\lambda^a\sigma^m\bar{\theta}, \tag{A.4}$$

$$\delta_\theta^{(1,\xi)} \lambda^a = \sigma^{mn}\theta v_{mn}^a + i\theta\left(\hat{D}^a - \sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(\xi A^{*0}\right)\right), \tag{A.5}$$

where $\hat{D}^a$ and $\hat{F}^a$ are given in terms of fermion bilinears by (2.30), (2.31). We have introduced the superscript $(1, \xi)$ to label the transformation fully. Operating $R$ from the left and $R^{-1}$ from the right on (A.5), we obtain

$$R\delta_\theta^{(1,\xi)}\lambda^a R^{-1} = \left(R\delta_\theta^{(1,\xi)} R^{-1}\right)\psi^a$$

$$= \sigma^{mn}\theta v_{mn}{}^a + i\theta\left(-\hat{D}^a - \sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(\xi A^{*0}\right)\right), \tag{A.6}$$

where we have used $R\hat{D}^a R^{-1} = -\hat{D}^a$. Eq. (A.6) is compared with $\boldsymbol{\delta}\psi^a$ at $\eta_1 = 0$ in (3.11) of the text, and we find

$$R\delta_{\eta_1=\theta}^{(1,\xi)} R^{-1} = \boldsymbol{\delta}_{\eta_1=0,\eta_2=\theta}^{(-\xi)} \equiv \delta_{\eta_2=\theta}^{(2,-\xi)} \tag{A.7}$$

on $\psi^a$. We have introduced the subscript and the superscript to $\boldsymbol{\delta}$ to specify the transformation completely. Proceeding in a similar way on (A.1), we obtain

$$R\delta^{(1,\xi)}\psi^a R^{-1} = R\delta^{(1,\xi)} R^{-1}(-\lambda^a)$$

$$= i\sqrt{2}\sigma^m\bar{\theta}\mathcal{D}_m A^a + \sqrt{2}\theta\left(\hat{F}^{*a} - \sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(eA^{*0} + m\mathcal{F}_0^*\right)\right). \tag{A.8}$$

24

We see that (A.7) is true on $\lambda^a$ as well. It is easy to check from (A.2), (A.4) that (A.7) holds on $A^a$ and on $v_m^a$. We conclude that (A.7) is valid on all fields.

Once this is established, it is immediate to provide a proof that our action is invariant under $\mathcal{N} = 2$ supersymmetry. Let

$$S(\xi) = \int d^4x \mathcal{L}(x) \ \text{ or } \ \int d^4x \mathcal{L}'(x), \tag{A.9}$$

where $\mathcal{L}(x)$ and $\mathcal{L}'(x)$ are given by (2.26) and by (2.33) respectively. $\mathcal{N} = 1$ supersymmetry implies

$$\delta_{\eta_1=\theta}^{(1,\xi)} S(\xi) = 0 \ . \tag{A.10}$$

Multiplying $R$ from left and $R^{-1}$ from right, we obtain

$$\left( R\delta_{\eta_1=\theta}^{(1,\xi)} R^{-1} \right) R S(\xi) R^{-1} = \delta_{\eta_2=\theta}^{(2,-\xi)} S(-\xi) = 0, \ \text{ and thus } \ \delta_{\eta_2=\theta}^{(2,\xi)} S(\xi) = 0, \tag{A.11}$$

which is a statement that our action is $\mathcal{N} = 2$ supersymmetric.


# Appendix  B

In this appendix, we reexamine the current supermultiplet in the Wess-Zumino model. While its superfield expression is well-known, we will present this supermultiplet in the component formalism, so that the reasoning here is applicable to the discussion in the text. The action is

$$S = \int d^4x \mathcal{L} \ , \ \mathcal{L} = \int d^2\theta d^2\bar{\theta} \Phi^* \Phi + \int d^2\theta W(\Phi) + \int d^2\bar{\theta} W^*(\Phi^*) \tag{B.1}$$

and the superpotential $W(\Phi)$ (or $W^*(\Phi^*)$) is a monomial of degree $k$ in $\Phi$ (or $\Phi^*$) . The model possesses $U(1)_R$ symmetry associated with

$$R: \quad \Phi(x,\theta,\bar{\theta}) \rightarrow e^{i\alpha/k}\Phi(x, e^{-i\alpha/2}\theta, \bar{\theta}), \tag{B.2}$$

so that

$$R: \quad A \rightarrow e^{i\alpha/k}A,$$
$$\psi \rightarrow e^{i\alpha(\frac{1}{k}-\frac{1}{2})}\psi, \tag{B.3}$$
$$F \rightarrow e^{i\alpha(\frac{1}{k}-1)}F.$$

The proper Noether current $J_{\alpha\dot{\alpha}}$ is given by

$$\theta J\bar{\theta} = \bar{\psi}\bar{\theta}\theta\psi + c\frac{i}{2}A^*\theta \overleftrightarrow{\partial} \cdot \sigma\bar{\theta}A \tag{B.4}$$
$$\equiv \theta j\bar{\theta} + c\theta \Delta j\bar{\theta}, \tag{B.5}$$

25

where

$$c = \frac{1}{1 - k/2} \tag{B.6}$$

in accordance with the $R$ weights of the fields which are read off from (B.3). We have introduced grassman coordinates $\theta^\alpha, \bar\theta_{\dot\alpha}$ to contract and suppress spinorial indices. The dot implies a contraction of Minkowski indices. The second term $(\Delta j)_{\alpha\dot\alpha}$ is known as the improvement term.

Let us check the supersymmetry transformation of (B.4), which acts as the lowest component of the supermultiplet;

$$
\begin{aligned}
\theta \delta j \bar\theta &\equiv \theta \eta^\alpha s_\alpha \bar\theta + \theta \bar\eta_{\dot\alpha} \bar s^{\dot\alpha} \bar\theta \\
&= \left( -i\sqrt{2}\eta\sigma\bar\theta \cdot \partial A^* + \sqrt{2}\theta\bar\eta F^* \right) \theta\psi + \bar\psi\bar\theta \left( i\sqrt{2}\theta\sigma\bar\eta \cdot \partial A + \sqrt{2}\theta\eta F \right), \quad \text{(B.7)}
\end{aligned}
$$

$$
\begin{aligned}
\theta \delta (\Delta j) \bar\theta &\equiv \theta \eta^\alpha (\Delta s)_\alpha \bar\theta + \theta \bar\eta_{\dot\alpha} (\Delta \bar s)^{\dot\alpha} \bar\theta \\
&= \frac{i}{2} A^* \theta\sigma \cdot \overleftrightarrow{\partial} \bar\theta (\sqrt{2}\eta\psi) + \frac{i}{2}(\sqrt{2}\bar\eta\bar\psi)\theta\sigma \cdot \overleftrightarrow{\partial} \bar\theta A. \tag{B.8}
\end{aligned}
$$

The improved supercurrents are

$$-\frac{1}{2} \operatorname{tr} \bar\sigma^m \left( s_\alpha + c(\Delta s)_\alpha \right) \quad \text{and} \quad -\frac{1}{2} \operatorname{tr} \bar\sigma^m \left( \bar s^{\dot\alpha} + c(\Delta \bar s)^{\dot\alpha} \right). \tag{B.9}$$

It is easy to check

$$\theta \left( \delta j + c\delta(\Delta j) \right) \bar\theta \Big|_{\eta=\theta, \ \bar\eta=\bar\theta} = 0 \tag{B.10}$$

if and only if $k = 3$ and therefore $c = -2$ from (B.6). This is nothing but the condition that the supercurrents (B.9) implement the superconformal constraints, that is, the irreducibility of their spin when the coupling constant in the superpotential is dimensionless.

Let us further transform (B.7) and (B.8) to generate the stress-energy tensor and we check the consistency with the supersymmetry algebra as well;

$$\theta \delta\delta j \bar\theta = (\bar\psi\bar\theta)(\theta\delta\delta\psi) + 2(\delta\bar\psi\bar\theta)(\theta\delta\psi) + (\delta\delta\bar\psi\bar\theta)(\theta\psi), \tag{B.11}$$

$$\theta \delta\delta (\Delta j) \bar\theta = \frac{i}{2} A^* \theta \overleftrightarrow{\partial} \cdot \sigma\bar\theta(\delta\delta A) + i\delta A^* \theta \overleftrightarrow{\partial} \cdot \bar\theta\delta A + \frac{i}{2}(\delta\delta A^*)\theta \overleftrightarrow{\partial} \cdot \sigma\bar\theta A. \tag{B.12}$$

The fermionic part of (B.11) is

$$
\begin{aligned}
(\bar\psi\bar\theta)(\theta\delta\delta\psi) + (\delta\delta\bar\psi\bar\theta)(\theta\psi) &= -2i(\theta\eta)(\bar\theta\bar\eta)\bar\psi\bar\sigma \cdot \overleftrightarrow{\partial} \psi + 2i(\bar\theta\bar\sigma\theta) \cdot \left( (\bar\psi\bar\eta) \overleftrightarrow{\partial} (\psi\eta) \right) \\
&\quad -2i(\bar\eta\bar\sigma\theta) \cdot \partial \left( (\bar\psi\bar\theta)(\eta\psi) \right) + 2i(\bar\theta\bar\sigma\eta) \cdot \partial \left( (\bar\psi\bar\eta)(\theta\psi) \right).
\end{aligned}
\tag{B.13}
$$

The bosonic part of (B.11) is

$$
\begin{aligned}
2(\delta\bar\psi\bar\theta)(\theta\delta\psi) &= 4(\eta\theta)(\bar\eta\bar\theta)(F^*F - \partial A^* \cdot \partial A) - 4(\theta\sigma\bar\theta) \cdot \partial A^*(\eta\sigma\bar\eta) \cdot \partial A \tag{B.14} \\
&\quad + 8(\eta\theta)(\bar\theta\bar\sigma^{mn}\bar\eta)\partial_m A^* \partial_n A - 2i(\bar\eta\bar\eta)F^*(\theta\sigma\bar\theta) \cdot \partial A + 2i(\eta\eta)(\theta\sigma\bar\theta) \cdot \partial A^* F.
\end{aligned}
$$

The fermionic part of (B.12) is

$$i\delta A^* \theta \overleftrightarrow{\partial} \cdot \bar\theta \delta A = 2i(\theta\sigma\bar\theta) \cdot (\bar\eta\bar\psi \overleftrightarrow{\partial} \eta\psi). \tag{B.15}$$

The bosonic part of (B.12) is

$$\frac{i}{2}A^* \theta \overleftrightarrow{\partial} \cdot \sigma\bar\theta(\delta\delta A) + \frac{i}{2}(\delta\delta A^*)\theta \overleftrightarrow{\partial} \cdot \sigma\bar\theta A$$
$$= -(\theta\sigma\bar\theta \cdot A^* \overleftrightarrow{\partial})(\partial A \cdot \eta\sigma\bar\eta) + (\eta\sigma\bar\eta \cdot \partial A^*)(\overleftrightarrow{\partial} A \cdot \theta\sigma\bar\theta)$$
$$+ i(\eta\eta)\theta\sigma\bar\theta \cdot A^* \overleftrightarrow{\partial} F + i(\bar\eta\bar\eta)(\theta\sigma\bar\theta) \cdot F^* \overleftrightarrow{\partial} A. \tag{B.16}$$

The consistency of the supersymmetry algebra demands that the $\eta\eta$ term and the $\bar\eta\bar\eta$ term be absent in $\theta\delta\delta J\bar\theta$. Let us check that this is in fact the case. From (B.14) and (B.16), we see that the $\bar\eta\bar\eta$ term is

$$-2i(\bar\eta\bar\eta)F^*(\theta\sigma\bar\theta) \cdot \partial A + ci(\bar\eta\bar\eta)(\theta\sigma\bar\theta) \cdot F^* \overleftrightarrow{\partial} A. \tag{B.17}$$

Using equation of motion for auxiliary fields $F$, $F^*$ and that $W(A)$ is a degree $k$ monomial in $A$, this is equal to

$$2i(\bar\eta\bar\eta)\left(1 - \frac{c}{2} + \frac{c}{2}(k-1)\right)\theta\sigma\bar\theta \cdot \partial A, \tag{B.18}$$

which vanishes when $c$ is chosen as (B.6).

The remainder of $\theta\delta\delta J\bar\theta$ closes into the stress-energy tensor. Using equations of motion, we have checked

$$\theta\delta\delta J\bar\theta = -2(c-1)T. \tag{B.19}$$

Here

$$T \equiv \eta\sigma^m\bar\eta\theta\sigma^n\bar\theta T_{mn}$$
$$= -(\eta\sigma\bar\eta) \cdot \partial A^*(\theta\sigma\bar\theta) \cdot \partial A - (\eta\sigma\bar\eta) \cdot \partial A(\theta\sigma\bar\theta) \cdot \partial A^*$$
$$-\frac{i}{2}\left(\theta\sigma\bar\theta \cdot \bar\eta\bar\psi \overleftrightarrow{\partial} \eta\psi + \eta\sigma\bar\eta \cdot \bar\theta\bar\psi \overleftrightarrow{\partial} \theta\psi\right) + 2\eta\theta\bar\eta\bar\theta\mathcal{L}. \tag{B.20}$$

# References

[1] J. Wess and J. Bagger, "Supersymmetry and supergravity," (second edition, Princeton University Press, 1992).

27

[2] S. Weinberg, "The quantum theory of fields. Vol. 3: Supersymmetry," (Cambridge University Press, 2000).

[3] M. F. Sohnius, "Introducing Supersymmetry," Phys. Rept. **128** (1985) 39.

[4] B. de Wit, J. W. van Holten and A. Van Proeyen, "Transformation Rules Of N=2 Supergravity Multiplets," Nucl. Phys. B **167** (1980) 186; J. Bagger and E. Witten, "Matter Couplings In N=2 Supergravity ," Nucl. Phys. B **222** (1983) 1; B. de Wit, P. G. Lauwers, R. Philippe, S. Q. Su and A. Van Proeyen, "Gauge And Matter Fields Coupled To N=2 Supergravity," Phys. Lett. B **134** (1984) 37; B. de Wit and A. Van Proeyen, "Potentials And Symmetries Of General Gauged N=2 Supergravity - Yang-Mills Models," Nucl. Phys. B **245** (1984) 89; B. de Wit, P. G. Lauwers and A. Van Proeyen, "Lagrangians Of N=2 Supergravity - Matter Systems," Nucl. Phys. B **255** (1985) 569; E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, "Vector Multiplets Coupled To N=2 Supergravity: Superhiggs Effect, Flat Potentials And Geometric Structure," Nucl. Phys. B **250** (1985) 385; H. Itoyama, L. D. McLerran, T. R. Taylor and J. J. van der Bij, "N=2 No Scale Supergravity," Nucl. Phys. B **279** (1987) 380; R. D'Auria, S. Ferrara and P. Fre, "Special And Quaternionic Isometries: General Couplings In N=2 Supergravity And The Scalar Potential," Nucl. Phys. B **359** (1991) 705; L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara and P. Fre', "General Matter Coupled N=2 Supergravity," Nucl. Phys. B **476** (1996) 397 [arXiv:hep-th/9603004]; L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre and T. Magri, "N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map," J. Geom. Phys. **23** (1997) 111 [arXiv:hep-th/9605032].

[5] S. Cecotti, L. Girardello and M. Porrati, "An Exceptional N=2 Supergravity With Flat Potential And Partial Superhiggs," Phys. Lett. B **168** (1986) 83; S. Ferrara, L. Girardello and M. Porrati, "Minimal Higgs Branch for the Breaking of Half of the Supersymmetries in N=2 Supergravity," Phys. Lett. B **366** (1996) 155 [arXiv:hep-th/9510074]; P. Fre, L. Girardello, I. Pesando and M. Trigiante, "Spontaneous N = 2 → N = 1 local supersymmetry breaking with surviving compact gauge groups," Nucl. Phys. B **493** (1997) 231 [arXiv:hep-th/9607032]; M. Porrati, "Spontaneous breaking of extended supersymmetry in global and local theories," Nucl. Phys. Proc. Suppl. **55B** (1997) 240 [arXiv:hep-th/9609073]; L. Andrianopoli, R. D'Auria and S. Ferrara, "Supersymmetry reduction of N-extended supergravities in four dimensions," JHEP **0203** (2002) 025 [arXiv:hep-th/0110277]; L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, "Super Higgs effect in extended supergravity," Nucl. Phys. B **640** (2002) 46 [arXiv:hep-th/0202116]; L. Andrianopoli, R. D'Auria and S. Ferrara, "Consistent reduction of N = 2 → N = 1 four dimensional supergravity coupled to matter," Nucl. Phys.

B **628** (2002) 387 [arXiv:hep-th/0112192]; J. Louis, "Aspects of spontaneous N = 2 → N = 1 breaking in supergravity," arXiv:hep-th/0203138; L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, "Gauging of flat groups in four dimensional supergravity," JHEP **0207** (2002) 010 [arXiv:hep-th/0203206]; L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, "Duality and spontaneously broken supergravity in flat backgrounds," Nucl. Phys. B **640** (2002) 63 [arXiv:hep-th/0204145]; L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, "N = 2 super-Higgs, N = 1 Poincare vacua and quaternionic geometry," JHEP **0301** (2003) 045 [arXiv:hep-th/0212236].

[6] R. Altendorfer and J. Bagger, "Dual supersymmetry algebras from partial supersymmetry breaking," Phys. Lett. B **460** (1999) 127 [arXiv:hep-th/9904213].

[7] J. R. David, E. Gava and K. S. Narain, "Partial N = 2 → N = 1 supersymmetry breaking and gravity deformed chiral rings," JHEP **0406** (2004) 041 [arXiv:hep-th/0311086].

[8] J. Hughes and J. Polchinski, "Partially Broken Global Supersymmetry And The Superstring," Nucl. Phys. B **278** (1986) 147; J. Hughes, J. Liu and J. Polchinski, "Supermembranes," Phys. Lett. B **180** (1986) 370.

[9] A. Achucarro, J. P. Gauntlett, K. Itoh and P. K. Townsend, "World Volume Supersymmetry From Space-Time Supersymmetry Of The Four-Dimensional Supermembrane," Nucl. Phys. B **314** (1989) 129.

[10] J. Bagger and A. Galperin, "A new Goldstone multiplet for partially broken supersymmetry," Phys. Rev. D **55** (1997) 1091 [arXiv:hep-th/9608177]; J. A. Bagger, "Partial breaking of extended supersymmetry," Nucl. Phys. Proc. Suppl. **52A** (1997) 362 [arXiv:hep-th/9610022]; J. Bagger and A. Galperin, "The tensor Goldstone multiplet for partially broken supersymmetry," Phys. Lett. B **412** (1997) 296 [arXiv:hep-th/9707061]; J. Bagger and A. Galperin, "Matter couplings in partially broken extended supersymmetry," Phys. Lett. B **336** (1994) 25 [arXiv:hep-th/9406217].

[11] M. Rocek and A. A. Tseytlin, "Partial breaking of global D = 4 supersymmetry, constrained superfields, and 3-brane actions," Phys. Rev. D **59** (1999) 106001 [arXiv:hep-th/9811232].

[12] Y. Gotoh and T. Uematsu, "Nonlinear realization of partially broken N = 2 superconformal symmetry in four dimensions," Phys. Lett. B **420** (1998) 69 [arXiv:hep-th/9707056].

[13] F. Gonzalez-Rey, I. Y. Park and M. Rocek, "On dual 3-brane actions with partially broken N = 2 supersymmetry," Nucl. Phys. B **544** (1999) 243 [arXiv:hep-th/9811130].

[14] E. Kiritsis and C. Kounnas, "Perturbative and non-perturbative partial supersymmetry breaking: N = 4 → N = 2 → N = 1," Nucl. Phys. B **503** (1997) 117

[arXiv:hep-th/9703059]; T. R. Taylor and C. Vafa, "RR flux on Calabi-Yau and partial supersymmetry breaking," Phys. Lett. B **474** (2000) 130 [arXiv:hep-th/9912152]; P. Mayr, "On supersymmetry breaking in string theory and its realization in brane worlds," Nucl. Phys. B **593** (2001) 99 [arXiv:hep-th/0003198]; G. Curio, A. Klemm, D. Lust and S. Theisen, "On the vacuum structure of type II string compactifications on Calabi-Yau spaces with H-fluxes," Nucl. Phys. B **609** (2001) 3 [arXiv:hep-th/0012213].

[15] S. Bellucci, E. Ivanov and S. Krivonos, "Partial breaking of N = 1 D = 10 supersymmetry," Phys. Lett. B **460** (1999) 348 [arXiv:hep-th/9811244]; E. Ivanov and S. Krivonos, "N = 1 D = 4 supermembrane in the coset approach," Phys. Lett. B **453** (1999) 237 [arXiv:hep-th/9901003]; S. Bellucci, E. Ivanov and S. Krivonos, "Superworldvolume dynamics of superbranes from nonlinear realizations," Phys. Lett. B **482** (2000) 233 [arXiv:hep-th/0003273]; S. Bellucci, E. Ivanov and S. Krivonos, "N = 2 and N = 4 supersymmetric Born-Infeld theories from nonlinear realizations," Phys. Lett. B **502** (2001) 279 [arXiv:hep-th/0012236]; S. Bellucci, E. Ivanov and S. Krivonos, "Towards the complete N = 2 superfield Born-Infeld action with partially broken N = 4 supersymmetry," Phys. Rev. D **64** (2001) 025014 [arXiv:hep-th/0101195].

[16] P. C. West, "Automorphisms, non-linear realizations and branes," JHEP **0002** (2000) 024 [arXiv:hep-th/0001216].

[17] S. V. Ketov, "A manifestly N = 2 supersymmetric Born-Infeld action," Mod. Phys. Lett. A **14** (1999) 501 [arXiv:hep-th/9809121]; S. V. Ketov, "N = 2 super-Born-Infeld theory revisited," Class. Quant. Grav. **17** (2000) L91 [arXiv:hep-th/0005126].

[18] S. M. Kuzenko and S. Theisen, "Supersymmetric duality rotations," JHEP **0003** (2000) 034 [arXiv:hep-th/0001068]; S. M. Kuzenko and S. Theisen, "Nonlinear self-duality and supersymmetry," Fortsch. Phys. **49** (2001) 273 [arXiv:hep-th/0007231].

[19] T. E. Clark, M. Nitta and T. ter Veldhuis, "Brane dynamics from non-linear realizations," Phys. Rev. D **67** (2003) 085026 [arXiv:hep-th/0208184]; T. E. Clark, M. Nitta and T. ter Veldhuis, "The effective action for brane gauge fields," Phys. Rev. D **69** (2004) 047701 [arXiv:hep-th/0209142].

[20] A. De Castro, L. Quevedo and A. Restuccia, "N = 2 Super-Born-Infeld from partially broken N = 3 supersymmetry in d = 4," JHEP **0406** (2004) 055 [arXiv:hep-th/0405062].

[21] for example see below and refs. thereof
M. Cvetic, F. Quevedo and S. J. Rey, "Stringy domain walls and target space modular invariance," Phys. Rev. Lett. **67** (1991) 1836; E. R. C. Abraham and P. K. Townsend, "Intersecting Extended Objects In Supersymmetric Field Theories," Nucl. Phys. B **351**

(1991) 313; M. Cvetic, S. Griffies and S. J. Rey, "Static domain walls in N=1 supergravity," Nucl. Phys. B **381** (1992) 301 [arXiv:hep-th/9201007]; M. Cvetic, S. Griffies and H. H. Soleng, "Local and global gravitational aspects of domain wall space-times," Phys. Rev. D **48** (1993) 2613 [arXiv:gr-qc/9306005]; G. R. Dvali and M. A. Shifman, "Dynamical compactification as a mechanism of spontaneous supersymmetry breaking," Nucl. Phys. B **504** (1997) 127 [arXiv:hep-th/9611213]; G. R. Dvali and M. A. Shifman, "Domain walls in strongly coupled theories," Phys. Lett. B **396** (1997) 64 [Erratum-ibid. B **407** (1997) 452] [arXiv:hep-th/9612128]; M. Naganuma and M. Nitta, "BPS domain walls in models with flat directions," Prog. Theor. Phys. **105** (2001) 501 [arXiv:hep-th/0007184]; M. Naganuma, M. Nitta and N. Sakai, "BPS walls and junctions in SUSY nonlinear sigma models," Phys. Rev. D **65** (2002) 045016 [arXiv:hep-th/0108179].

[22] I. Antoniadis, H. Partouche and T. R. Taylor, "Spontaneous Breaking of N=2 Global Supersymmetry," Phys. Lett. B **372** (1996) 83 [arXiv:hep-th/9512006]; I. Antoniadis and T. R. Taylor, "Dual N=2 SUSY Breaking," Fortsch. Phys. **44** (1996) 487 [arXiv:hep-th/9604062].

[23] R. Grimm, M. Sohnius and J. Wess, "Extended Supersymmetry And Gauge Theories," Nucl. Phys. B **133** (1978) 275; M. F. Sohnius, "Bianchi Identities For Supersymmetric Gauge Theories," Nucl. Phys. B **136** (1978) 461; M. de Roo, J. W. van Holten, B. de Wit and A. Van Proeyen, "Chiral Superfields In N=2 Supergravity," Nucl. Phys. B **173** (1980) 175.

[24] S. Ferrara, L. Girardello and M. Porrati, "Spontaneous Breaking of N=2 to N=1 in Rigid and Local Supersymmetric Theories," Phys. Lett. B **376** (1996) 275 [arXiv:hep-th/9512180]; H. Partouche and B. Pioline, "Partial spontaneous breaking of global supersymmetry," Nucl. Phys. Proc. Suppl. **56B** (1997) 322 [arXiv:hep-th/9702115].

[25] E. A. Ivanov and B. M. Zupnik, "Modified N = 2 supersymmetry and Fayet-Iliopoulos terms," Phys. Atom. Nucl. **62** (1999) 1043 [Yad. Fiz. **62** (1999) 1110] [arXiv:hep-th/9710236].

[26] P. Fayet and J. Iliopoulos, "Spontaneously Broken Supergauge Symmetries And Goldstone Spinors," Phys. Lett. B **51** (1974) 461; P. Fayet, "Fermi-Bose Hypersymmetry," Nucl. Phys. B **113** (1976) 135.

[27] J. T. Lopuszanski, "The Spontaneously Broken Supersymmetry In Quantum Field Theory," Rept. Math. Phys. **13** (1978) 37.

[28] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," Nucl. Phys. B **644** (2002) 3 [arXiv:hep-th/0206255]; R. Dijkgraaf and C. Vafa, "On geometry and matrix models," Nucl. Phys. B **644** (2002) 21 [arXiv:hep-th/0207106]; R. Dijkgraaf and C. Vafa, "A perturbative window into nonperturbative physics," arXiv:hep-th/0208048.

[29] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," Nucl. Phys. B **426** (1994) 19 [Erratum-ibid. B **430** (1994) 485] [arXiv:hep-th/9407087].

[30] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, "Simple singularities and N=2 supersymmetric Yang-Mills theory," Phys. Lett. B **344** (1995) 169 [arXiv:hep-th/9411048]; P. C. Argyres and A. E. Faraggi, "The vacuum structure and spectrum of N=2 supersymmetric SU(n) gauge theory," Phys. Rev. Lett. **74** (1995) 3931 [arXiv:hep-th/9411057]; A. Hanany and Y. Oz, "On the quantum moduli space of vacua of N=2 supersymmetric SU(N(c)) gauge theories," Nucl. Phys. B **452** (1995) 283 [arXiv:hep-th/9505075]; M. Matone, "Instantons and recursion relations in N=2 SUSY gauge theory," Phys. Lett. B **357** (1995) 342 [arXiv:hep-th/9506102]; G. Bonelli, M. Matone and M. Tonin, "Solving N = 2 SYM by reflection symmetry of quantum vacua," Phys. Rev. D **55** (1997) 6466 [arXiv:hep-th/9610026].

[31] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, "Integrability and Seiberg-Witten exact solution," Phys. Lett. B **355** (1995) 466 [arXiv:hep-th/9505035]; E. J. Martinec and N. P. Warner, "Integrable systems and supersymmetric gauge theory," Nucl. Phys. B **459** (1996) 97 [arXiv:hep-th/9509161]; T. Nakatsu and K. Takasaki, "Whitham-Toda hierarchy and N = 2 supersymmetric Yang-Mills theory," Mod. Phys. Lett. A **11** (1996) 157 [arXiv:hep-th/9509162]; R. Donagi and E. Witten, "Supersymmetric Yang-Mills Theory And Integrable Systems," Nucl. Phys. B **460** (1996) 299 [arXiv:hep-th/9510101]; T. Eguchi and S. K. Yang, "Prepotentials of $N = 2$ Supersymmetric Gauge Theories and Soliton Equations," Mod. Phys. Lett. A **11** (1996) 131 [arXiv:hep-th/9510183]; H. Itoyama and A. Morozov, "Integrability and Seiberg-Witten Theory: Curves and Periods," Nucl. Phys. B **477** (1996) 855 [arXiv:hep-th/9511126]; H. Itoyama and A. Morozov, "Integrability and Seiberg-Witten theory," arXiv:hep-th/9601168; H. Itoyama and A. Morozov, "Prepotential and the Seiberg-Witten Theory," Nucl. Phys. B **491** (1997) 529 [arXiv:hep-th/9512161].

[32] L. Chekhov and A. Mironov, "Matrix models vs. Seiberg-Witten/Whitham theories," Phys. Lett. B **552** (2003) 293 [arXiv:hep-th/0209085]; H. Itoyama and A. Morozov, "The Dijkgraaf-Vafa prepotential in the context of general Seiberg-Witten theory," Nucl. Phys. B **657** (2003) 53 [arXiv:hep-th/0211245]; H. Itoyama and A. Morozov,

"Experiments with the WDVV equations for the gluino-condensate prepotential: The cubic (two-cut) case," Phys. Lett. B **555** (2003) 287 [arXiv:hep-th/0211259]; H. Itoyama and A. Morozov, "Calculating gluino condensate prepotential," Prog. Theor. Phys. **109** (2003) 433 [arXiv:hep-th/0212032]; M. Matone, "Seiberg-Witten duality in Dijkgraaf-Vafa theory," Nucl. Phys. B **656** (2003) 78 [arXiv:hep-th/0212253]; L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, "DV and WDVV," Phys. Lett. B **562** (2003) 323 [arXiv:hep-th/0301071]; A. Dymarsky and V. Pestun, "On the property of Cachazo-Intriligator-Vafa prepotential at the extremum of the superpotential," Phys. Rev. D **67** (2003) 125001 [arXiv:hep-th/0301135]; H. Itoyama and A. Morozov, "Gluino-condensate (CIV-DV) prepotential from its Whitham-time derivatives," Int. J. Mod. Phys. A **18** (2003) 5889 [arXiv:hep-th/0301136]; S. Aoyama and T. Masuda, "The Whitham deformation of the Dijkgraaf-Vafa theory," JHEP **0403** (2004) 072 [arXiv:hep-th/0309232]; H. Itoyama and H. Kanno, "Whitham prepotential and superpotential," Nucl. Phys. B **686** (2004) 155 [arXiv:hep-th/0312306].

[33] H. Itoyama, M. Koike and H. Takashino, "N = 2 supermultiplet of currents and anomalous transformations in supersymmetric gauge theory," Mod. Phys. Lett. A **13** (1998) 1063 [arXiv:hep-th/9610228].

[34] P. Fre, "Lectures on Special Kähler Geometry and Electric–Magnetic Duality Rotations," Nucl. Phys. Proc. Suppl. **45BC** (1996) 59 [arXiv:hep-th/9512043]; A. Van Proeyen, "Vector multiplets in N = 2 supersymmetry and its associated moduli spaces," arXiv:hep-th/9512139; B. Craps, F. Roose, W. Troost and A. Van Proeyen, "What is special Kähler geometry?," Nucl. Phys. B **503** (1997) 565 [arXiv:hep-th/9703082]; D. S. Freed, "Special Kähler manifolds," Commun. Math. Phys. **203** (1999) 31 [arXiv:hep-th/9712042].

[35] J. Bagger and E. Witten, "The Gauge Invariant Supersymmetric Nonlinear Sigma Model," Phys. Lett. B **118** (1982) 103; C. M. Hull, A. Karlhede, U. Lindstrom and M. Rocek, "Nonlinear Sigma Models And Their Gauging In And Out Of Superspace," Nucl. Phys. B **266** (1986) 1.

[36] S. Ferrara and B. Zumino, "Transformation Properties Of The Supercurrent," Nucl. Phys. B **87** (1975) 207.

[37] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, "Chiral rings and anomalies in supersymmetric gauge theory," JHEP **0212** (2002) 071 [arXiv:hep-th/0211170].

[38] H. Itoyama and H. Kanno, "Supereigenvalue model and Dijkgraaf-Vafa proposal," Phys. Lett. B **573** (2003) 227 [arXiv:hep-th/0304184].